Relations between the degrees of transitive constituents of \( G_1 \) and the absolutely irreducible constituents of permutation representation \( G^* \) of the group \( G \)

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Abstract

Let \( \Omega \) be a finite set of arbitrary elements, \( G \) be permutation group on \( \Omega \). \( \Delta \subseteq \Omega \), \( G_1 = \{1\} = \Delta_1, \Delta_2, \ldots, \Delta_n \) are \( n \) orbits of \( G \), \( n_i = |\Delta_i| \), \( f_i \) is the degree of different irreducible representation of \( D_1 \ldots D_r \) appearing in \( G^* \), \( V = V(G) \) be the ring of all the matrices of \( G \), \( \dim V = k \). And also if the irreducible constituents of permutation representation \( G^* \) are all different, then the expression

\[
q = n^{k-2} \prod_{i=1}^{k} \frac{n_i}{\prod_{i=1}^{k} f_i^{e_i}}
\]

is a rational integer.

Keywords: irreducible constituents orbits, transitive constituents, permutation representation degree.


1. Introduction

In 1943, R. Brauer studied about permutation groups and find the permutation groups of prime degree and related classes of Groups (See) \([4]\). In 1928, M.J. Weiss studied the primitive groups which contain substitutions of prime order \( p \) and of degree \( 6p \) or \( 7p \), and in 1934 he worked on simply transitive groups and obtained the beautiful consequences, (See) \([25, 26]\). In 1906, W. Burnside introduced d researched about transitive groups, of prime degree (See) \([2]\), and in 1921, he worked about the certain simply- transitive permutation group and obtained a beautiful consequences (See) \([3]\).

From years 1906 to 1936, W.A. Manning studied about primitive groups and finding the primitive groups of classes six, ten, twelve and fifteen. (See) \([11-13]\). In 1937 J.S. Frame determined the degrees of the irreducible components of simply transitive permutation groups, and in 1941, he obtained the double cosets of a finite groups (See) \([8, 9]\). And also in 1952, he finding the irreducible representation extracted from two permutation groups (See) \([7]\). G.A. Miller (1897 & 1915), (See) \([19, 20]\), E.T. Parker (1954), (See) \([21]\), M. Suzuki (1962), (See) \([22]\), J.G. Thompson (1959), (See) \([23]\), M.J. Weiss (1928), (See) \([25, 26]\), H. Wielandt (1935 & 1956) (See) \([27, 28]\) and H. Zassenhaus (1935), (See) \([30]\) are studied about transitive and primitive groups and their obtained the beautiful and more consequence. In 1935, N. Itô, studied on primitive permutation groups and in 1962 he obtained transitive simple permutation groups of degree \( 2p \), (See) \([9, 10]\).

Now in this paper we will study the relations between the degrees \( n_i \) of the transitive constituents of \( G_1 \) and the degrees \( f_i \) of the absolutely irreducible constituents \( D_i \) of the permutation representation \( G^* \) of the group \( G \), and the end we expressed this relations in expression

\[
q = n^{k-2} \prod_{i=1}^{k} \frac{n_i}{\prod_{i=1}^{k} f_i^{e_i}}
\]

and also our prove that the expression \( q \) is a rational integer.

2. Preliminaries: In this chapter we study the elementary properties, lemmas and theorems, whose we will used the next chapter.
2.1. Elementary notions and definitions: Let $\Omega$ be a finite set of arbitrary elements which for natural numbers 1,2,...,n as the points and $\Delta$ subset of $\Omega$. Then a permutation on $\Omega$ is a one-to-one mapping of $\Omega$ onto itself. We denote the image of the point $\alpha \in \Omega$ under the permutation $p$ by $\alpha^p$. We write $P = \left( \begin{array}{cccc} 1 & 2 & \cdots & n \\ 1^p & 2^p & \cdots & n^p \end{array} \right) = \left( \begin{array}{c} \alpha \\ p \end{array} \right)$. We define the product $pq$ of two permutations $p$ and $q$ on $\Omega$ by the formula $a^{pq} = (a^p)^q$. Trivially, $pq$ is again a permutation on $\Omega$. Obviously $G$ is isomorphic to $G$. We call $G$ the permutation group of $G$. If this map $p \mapsto q$ is surjective, then each permutation in $G$ (namely, the one in $G^\Delta$) is transitive, that is, $G^\Delta \cong G$. We define the product $pq$ of two permutations $p$ and $q$ on $\Omega$. Then each $p \in G$ gives $1$ in this position. If $G$ is a permutation group on $\Omega$, in short $G \leq S^n$. We say that a set $\Delta \subseteq \Omega$ is a fixed block of $G$ or is fixed by $G$ if $\Delta = G^\Delta$. Then each $\beta \in G$ induces a permutation on $\Delta$ which we denote by $\beta^\Delta$. We call the totality of $G^\Delta$'s formed for all $\beta \in G$ the constituent $G^\Delta$ of $G$ on $\Delta$ (for example $G = G^\Delta$). $G^\Delta$ is a permutation group on $\Delta$. Obviously the mapping $G \mapsto G^\Delta$ is a homomorphism : $G \rightarrow G^\Delta$. If this mapping is an isomorphism, that is, $|G^\Delta| = |G|$, then the constituents $G^\Delta$ of $\Delta$ is called faithful. Every group $G$ on $\Omega$ has the trivial fixed blocks $\phi$ on $\Omega$. If it has no others it is called transitive. Otherwise it is called intransitive. Accordingly, a constituent $G^\Delta$ is transitive precisely when $\Delta$ is a minimal fixed block ($\Delta \neq \phi$). In this case $\Delta$ is called an orbit or set of transitivity of $G$. Every permutation $p$ on $\Omega$ can be regarded in the following way as a linear substitution in $|\Omega| = n$ variables. The variables $X_1,...,X_n$ are taken as points. We form column vectors $x = \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right]$, $x^p = \left[ \begin{array}{c} x_1^p \\ \vdots \\ x_n^p \end{array} \right] = g^x$ where $g^x = \left( \delta^x_{\alpha,\beta} \right)_{\alpha,\beta \in \Delta}$ is the $n$ by $n$ matrix corresponding to the linear transformation $x \mapsto x^p$. $(\delta^x_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$ is the well known Kronecker symbol). We call $g^x$ the permutation matrix corresponding to $g$. Such a matrix contains exactly one $1$ in each row and column and zeros everywhere else. In addition, every permutation matrix $g^x$ is orthogonal, i.e., the transpose $g^x$ of $g^x$ is identical with its inverse: $g^{x^t} = g^x$. We obtain a faithful representation of $G$ by $g \mapsto g^x$. Now let $G \leq S^n$. By $G^\Delta$ we denote the group of all matrices $g^x$ with $g \in G$. Obviously $G^x$ is isomorphic to $G$. We call $G^x$ the permutation representation of $G$.

2.2. Theorem (See [22]) If a transitive permutation group $G$ is regarded as a matrix group $G^x$, then the matrices which commute with all the matrices of $G^x$ form a ring $\mathbb{V}(G)$. We call $\mathbb{V}$ "the centralizer ring corresponding to $G"$. $\mathbb{V}$ is a vector space over the complex number field which has the matrices $B(\Delta)$ corresponding to the orbits $\Delta$ of $G$ as a linear basis. In particular, the dimension of $\mathbb{V}$ coincides with the number $k$ of orbits of $G_1$.

Proof: See [29, Theorem 28.4]

Let $D_1,...,D_r$ be the different irreducible representations appearing in $G^x$ where $D_1$ is the identity representation. In the following we denote by $f_i$ the degree of $D_i$ ($i=1,...,r$), and by $e_i$ the multiplicity of $D_i$ in $G^x$. In particular, we have $e_i f_i = n$, the reduction of $G^x$ gives for an appropriately chosen unitary $n$ by $n$ matrix $U$:

$$U^t G^x U = \left[ \begin{array}{c} D_1 \\ D_2 \\ \vdots \\ D_r \end{array} \right] .$$

2.3. Proposition: $\text{Dim} \, (\mathbb{V}) = \sum_{i=1}^r e_i f_i^2 = k$.

Proof: See [29, Proposition 29.2]

2.4. Definition: Let $G$ be transitive and consider the orbits of $G_1$. With each of these orbits $\Delta$ (including the trivial $\Delta = \{1\}$) we associate in the following way a matrix $B(\Delta) = \left( V^\Delta_{\alpha,\beta} \right)$, $\alpha, \beta = 1,...,n$, with elements:

$$V^\Delta_{\alpha,\beta} = \begin{cases} 1, & \text{if there exists } g \in G \text{ and } \delta \in \Delta \text{ with } l^g = \beta \text{ and } \delta^g = \alpha \\ 0, & \text{otherwise} \end{cases}$$

Thus, in the first column of $B(\Delta)$ we have exactly those $V^\Delta_{\alpha,\beta} = 1$ for which $\alpha \in \Delta$ holds. If $\Gamma \neq \Delta$, the ones of $B(\Gamma)$ and $B(\Delta)$ do not occur in the same place. On the other hand, for each place $(\alpha, \beta)$ there is an orbit $\Lambda$ of $G$, namely, the one in which the $\alpha^\beta$ with $l^\beta = \beta$ lies such that $B(\Lambda)$ has 1 in this position.
2.5. Proposition: Let M be the n by n matrix whose elements are all 1.then $\sum_{\Gamma} B(\Delta) = M$. (Here the summation is over all orbits of G)

Proof: See (29), proposition 28.2.

2.6. Theorem. V is commutative if and only if all the $e_i=1$.

Proof: See (29), Theorem 29.3

2.7. Theorem. V is commutative if and only if the class matrices $E_i = \sum_{g \in C_i} g$ (i = 1,...,n) whose $C_i$ be the ith class of conjugate elements of G, generate V, i.e., when each $B \in V$ has a (not necessarily unique) representation $B = \sum_{i=1}^{r} z_i E_i$.

Proof: See (29), Theorem 29.8

2.8. Theorem: Let $s, \Delta$ be two orbits of $G_i$, then $Tr(B(s)B(\Delta)) = \left\{ \begin{array}{ll}
\circ & \Gamma \neq \Delta \\
|\Gamma|n & \Gamma = \Delta
\end{array} \right.$

Proof: See (29), Theorem 28.10

3. Main result

In this section by used of the theorems and lemmas of section 2, we prove that the following main theorem:

3.1. Main theorem: Let $\Omega$ be a finite set of arbitrary elements, G be permutation group on $\Omega, \Delta \subseteq \Omega$, $G_1 = \{1\} = \Delta_1, \Delta_2, ..., \Delta_n$ are n orbits of G. $n_i = |\Delta_i|, f_i$ is the degree of different irreducible representation of $D_1,...,D_i$ appearing in $G^*$, $V=V(G)$ be the ring of all the matrices of $G$, dim$V=k$. And also if the irreducible constituents of permutation representation $G^*$ are all different, then $q = n^{k-2} \sum_{i=1}^{k} n_i / \sum_{i=1}^{k} f_i$ is a rational integer.

Before from proof of the main theorem, we first the prove the following theorems. And the end we consequence the main theorem.

3.2. Theorem: (A) If the irreducible constituents of $G^*$ are all different, i.e., if all the multiplicities $e_i=1$, then the rational number $q = n^{k-2} \prod_{i=1}^{k} n_i / \sum_{i=1}^{k} f_i$ is an integer.

(B) If in addition the k numbers $n_i$ are all different, then q is a square.

(C) If the irreducible constituents of $G^*$ all have rational characters, then q is a square. The hypothesis is always fulfilled if the degrees $f_i$ are all different.

Notice: It is not known whether (B) is true for $k>3$.

Proof of 3.2.(A). It suffices to show that q is an algebraic integer. The notation of the preceding section is continued.

Let U again be the unitary transformation matrix introduced in §2. From the hypothesis $e_i=1$ it follows that every matrix $M=U^{-1}BU$ with $B \in V$ has diagonal form. Because $B_i = B(\Delta_i) \in V(G)$ we have in particular $M_i = U^{-1}BU = [w_1, w_2, \mathcal{J}_{i_1}, ..., w_k, \mathcal{J}_{i_k}]$, which $\mathcal{J}_{i}$ is the $f_i$ by $f_i$ identity matrix.

Let $w_i$ be the diagonal elements of the matrix $M = U^{-1}BU$, for arbitrary $B \in V$. We put

$B = \sum_{i} z_i B_i, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix}, \quad N = \begin{array}{l} [1, n_2, ..., n_k], \\
F = [1, f_2, ..., f_k], \quad \text{and} \quad I=(w_i); \quad i,j=1, ..., k. \end{array}$

From $M = U^{-1}BU$ it follows that $\overline{M}M = U^{-1}\overline{B}^TBU$, since U was assumed unitary. With the aid of 2.8 we now obtain

$z^T N n z = \sum_{i=1}^{k} \overline{z}_i z_n w_i = \sum_{i,j} \overline{z}_i z_j Tr(B_i^T B_j) = Tr(\overline{B}^T B) = Tr(\overline{M}M) = \sum_{i} \overline{w}_i w_i f_i = w^T w.$

Because $w_i = \sum j z_j w_{ij}$, i.e., $w = I z$, we therefore have $N n = I F I$. By taking the determinant we get $n^{k} \prod_{i} n_i = |\overline{F}| |\overline{I}| |\prod_{i} f_i| |\overline{I}| |\overline{I}|$.
The \( w_{ij} \), as eigenvalues of the matrix \( B_i \) which has integer coefficients, are algebraic integers, and therefore \(|I|\) and \( |\tilde{I}| \) are also algebraic integers.

We wish to show that \(|I|\) is divisible by \( n \). by 2.5, \( \sum B_j = M \) where \( M \) is the \( n \times n \) matrix consisting of \( n^2 \) ones. \( M \) has the eigenvalue \( n \) occurring with multiplicity 1 belonging to the eigenvector \( \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \). Its remaining eigenvalues are 0. In the diagonal matrix \( \sum j M_j, n \) therefore appears exactly once, the remaining elements being 0. Therefore \( \sum j w_{ij} = n \) for \( i = 1 \) and \( = 0 \) for the remaining. This implies

\[
\begin{vmatrix}
\begin{array}{ccc}
 n & w_{12} & \cdots & w_{1k} \\
 0 & w_{22} & \cdots & w_{2k} \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & w_{k2} & \cdots & w_{kk}
\end{array}
\end{vmatrix} = 0 \pmod{n}.
\]

Hence \( q = n^{-2} |I| \) is an algebraic integer, hence also a rational integer.

**Proof of 3.2(C).** (a) Because of the hypothesis \( e_1 = \cdots = e_k = 1 \), the commutativity of \( V \) follows by 2.4. Theorem 2.7 yields the existence of \( k \) class matrices \( C_1, \ldots, C_k \) and of complex numbers \( x_{ij} \) such that \( B_i = \sum_{j=1}^{k} x_{ij} C_j \) \( (i = 1, \ldots, k) \). Conversely by 2.2 there are also \( x_{ij}' \) with \( C_i = \sum_{j=1}^{k} x_{ij}' B_j \). The \( x_{ij} \) are, by well-known theorems of linear algebra, uniquely determined and rational, since the matrices \( B_i \) and \( C_j \) are rational.

(b) By hypothesis all the irreducible characters appearing in \( G^* \) are rational. Thus the matrices \( U^{-1} C_u U \) appearing in the proof of 2.6 are also rational. By (a) the matrices \( M_i = U^{-1} B_i U = \sum x_{ij} U^{-1} C_j U \) are then rational. The \( w_{ij} \) are therefore rational. Since the \( w_{ij} \) were already in the proof of 3.2 (A) shown to be algebraic integers, they are rational integers. \(|I| = |w_{ij}| \) is therefore a rational integer. Since \( n \) divides \(|I|, n^{-1}|I| \) is also a rational integer, and \( q \) is therefore a square as was asserted.

(c) The hypothesis that all the irreducible characters appearing in \( G^* \) are rational is fulfilled if the degrees \( f_i \) of the irreducible constituents of \( G^* \) are all different. For since \( G^* \) is rational, with each irreducible representation \( D_i \) all representations conjugate to it appear in \( G^* \). Because all of the \( f_i \) are different, these coincide with \( D_i \), and \( X_i \) is therefore rational.

We assume in the sequel that \( V \) is commutative, i.e., that the irreducible constituents of \( G^* \) are all different, and draw some conclusions from 3.2.

**Theorem 3.3.** If the nontrivial irreducible representations appearing in \( G^* \) all have the same degree \( f \), then the orbits of \( G_1 \) different from 1 all have the same length, which also equals \( f \). Hence \( G \) is either regular and Abelian \((f = 1)\) or \((3/2)\)-fold transitive \((f > 1)\).

**Proof.** From \( f_i = f \) for \( i = 2, \ldots, k \) and 3.2(A) it follows that \( f^{k-1} \) divides \( n^{k-2} \prod_i n_i \). Since \( f \) divides \( n-1 \), \( f \) and \( n \) are relatively prime, hence \( f^{k-1} \) divides \( \prod_i n_i \). In addition we have

\[
(k - 1)f = n - 1 = \sum_{i=2}^{k} n_i,
\]

i.e.,

\[
\frac{1}{k - 1} \sum_{i=2}^{k} n_i = f.
\]

The geometric mean of the \( n_i \) is at most equal to the arithmetic mean, thus \( \prod_i n_i \leq f^{k-1} \). With the help of the divisibility property it now follows that \( \prod_i n_i = f^{k-1} \). The geometric and arithmetic mean of the \( n_i \) therefore coincide. This occurs, however, only if \( n_i = f \), as was to be proved.

From theorem 3.3 it follows immediately that:
Theorem 3.4. If \( k=3 \) and \( n_2 \neq n_3 \), then also \( f_2 / f_3 \). Hence 3.1 (B) holds for \( k=3 \).

It is not known if in general in general when the \( n_i \) are all different \((i=2, ..., k)\) it follows that the \( f_i \) are also all different. Should this prove correct, the conjecture 3.2(B) would be true.

In the proof of theorem 3.2 we used the matrix equation \( N \prod f_i = \prod f_i \) only to compare the determinants. More precise information can be obtained from the elementary divisors. Since the \( k \)-th elementary divisor of a product of matrices is known to be divisible by the \( k \)-th elementary divisor of each factor, the \( k \)-th elementary divisor of \( N \) is divisible by the \( k \)-th elementary divisor of \( F \). In other words:

**Theorem 3.5.** If, under the assumptions of 3.2(A), there are \( l \) of the numbers \( f_i \) divisible by a given prime power \( p^a \) then there are at least \( l \) of the numbers \( n_i \) divisible by \( p^a \).

Now by used from theorems 3.2, 3.3, 3.4 and 3.5 and as generalization of theorem 3.2 to the case where some multiplicities \( e_i > 1 \), we follows that:

**Main theorem 3.6.** The expression \( q = \prod_{i=1}^{k-2} n_i \prod_{i=1}^{k} f_i^{e_i} \) is a rational integer.

If in addition the irreducible constituents of \( G^* \) have rational characters, then \( q \) is square. The hypothesis is always satisfied if the degrees of the different irreducible constituents of \( G^* \) are all different.

Reference

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