Some Properties of Rw-Locally Closed Sets In Topological Spaces

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Abstract
In this paper, we introduce three weaker forms of locally closed sets called RW-LC sets, RW-LC* set and RW-LC** sets each of which is weaker than locally closed set and study some of their properties in topological spaces.

Keywords: Rw-closed sets, rw-open sets, locally closed sets, rw-locally closed sets

1. Introduction
Kuratowski and Sierpinski \[11\] introduced the notion of locally closed sets in topological spaces. According to Bourbaki \[6\], a subset of a topological space \((X, \tau)\) is locally closed in \((X, \tau)\) if it is the intersection of an open set and a closed set in \((X, \tau)\). Stone \[2\] has used the term FG for locally closed set. Ganster and Reilly \[7\] have introduced locally closed sets, which are weaker forms of both closed and open sets. After that Balachandran et al. \[2, 3\], Gnanambal \[10\], Arockiarani et al. \[1\], Pusphalatha \[12\] and Sheik John \[13\] have introduced \(\alpha\)-locally closed, generalized \(\alpha\)-locally closed, semi locally closed, semi generalized locally closed, regular generalized locally closed, strongly locally closed and \(w\)-locally closed sets and their continuous maps in topological spaces respectively. Recently as a generalization of closed sets \(w\)-closed sets and continuous maps were introduced and studied by Benchalli et al. \[12\]. A subset \(A\) of a topological space \((X, \tau)\) is said to be \(w\)-closed set if \(\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi regular-open.

2. Preliminaries: A subset \(A\) of topological space \((X, \tau)\) is called a

1. Locally closed (briefly LC) set \[7\] if \(A = U \cap F\), where \(U\) is open and \(F\) is closed in \(X\).
2. \(w\)-open set \[14\] if \(\text{Cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi regular-open.
3. \(w\)-closed set \[11\] if \(\alpha \text{Cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is \(w\)-open.
4. \(\theta\)-\(g\)-lc set \[4\] if \(A = U \cap F\), where \(U\) is \(\theta\)-\(g\)-open and \(F\) is \(\theta\)-\(g\)-closed in \(X\).
5. \(\theta\)-\(g\)-lc* set \[4\] if \(A = U \cap F\), where \(U\) is \(\theta\)-\(g\)-open and \(F\) \(\theta\)-\(g\)-closed in \(X\).
6. \(\theta\)-\(g\)-lc set** \[4\] if \(A = U \cap F\), where \(U\) is \(\theta\)-\(g\)-open and \(F\) \(\theta\)-\(g\)-closed in \(X\).
7. \(g\)-lc set \[4\] if \(A = U \cap F\), where \(U\) is \(g\)-open and \(F\) is \(g\)-closed in \(X\).
8. \(g\)-lc* set \[4\] if \(A = U \cap F\), where \(U\) is \(g\)-open and \(F\) \(g\)-closed in \(X\).
9. \(g\)-lc set** \[4\] if \(A = U \cap F\), where \(U\) is \(g\)-open and \(F\) \(g\)-closed in \(X\).
10. \(w\)-lc set \[8\] if \(A = U \cap F\), where \(U\) is \(w\)-open and \(F\) is \(w\)-closed in \(X\).
11. \(w\)-lc* set \[8\] if \(A = U \cap F\), where \(U\) is \(w\)-open and \(F\) \(w\)-closed in \(X\).
12. \(w\)-lc** set \[8\] if \(A = U \cap F\), where \(U\) is \(w\)-open and \(F\) \(w\)-closed in \(X\).
13. \(rg\)-lc set \[5\] if \(A = U \cap F\), where \(U\) is \(rg\)-open and \(F\) \(rg\)-closed in \(X\).
14. \(rg\)-lc* set \[5\] if \(A = U \cap F\), where \(U\) is \(rg\)-open and \(F\) \(rg\)-closed in \(X\).
15. \(rg\)-lc** set \[5\] if \(A = U \cap F\), where \(U\) is \(rg\)-open and \(F\) \(rg\)-closed in \(X\).
16. $\delta g$-lc set if $A = U \cap F$ (1) where $U$ is $I_\delta g$-open and $F$ is $I_\delta g$-closed in $X$.
17. $I_\delta g$-lc* set if $A = U \cap F$ (3) where $U$ is $I_\delta g$-open and $F$ is $I_\delta g$-closed in $X$.
18. $I_\delta g$-lc** set if $A = U \cap F$ (3) where $U$ is open and $F$ is $I_\delta g$-closed in $X$.
19. $\alpha r\omega$-lc set if $A = U \cap F$ (14) where $U$ is $\alpha r\omega$-open and $F$ is $\alpha r\omega$-closed in $X$.
20. $\alpha r\omega$-lc* set if $A = U \cap F$ (14) where $U$ is $\alpha r\omega$-open and $F$ is closed in $X$.
21. $\alpha r\omega$-lc** set if $A = U \cap F$ (14) where $U$ is open and $F$ is $\alpha r\omega$-closed in $X$.

2.1 Lemma: (14)
(i) Every closed set is $r\omega$-closed set.
(ii) Every $w$-closed set is $r\omega$-closed set.
(iii) Every $\theta$-closed set is $r\omega$-closed set.
(iv) Every $\delta_\gamma$-closed set is $r\omega$-closed set.
(v) Every $\delta_\gamma$-closed set is $\alpha r\omega$-closed set.

2.2 Lemma: The space $(X, \tau)$ is $T_{\alpha r\omega}$-space if every $r\omega$-closed set is closed set.

3. $r\omega$-locally closed sets in topological spaces

3.1 Definition: A Subset $A$ of t.s $(X, \tau)$ is called $r\omega$-locally closed (briefly $r\omega$-LC) if $A = U \cap F$ where $U$ is $r\omega$-open in $(X, \tau)$ and $F$ is $r\omega$-closed in $(X, \tau)$.

The set of all $r\omega$-locally closed sets of $(X, \tau)$ is denoted by $r\omega$-LC $(X, \tau)$.

3.2 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$
$\mathcal{R}(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \{b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, c, d\}\}$.
$\mathcal{R}(X, \tau) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, c, d\}\}.$

3.3 Remark: The following are well known
(i) A Subset $A$ of $(X, \tau)$ is $r\omega$-LC set iff it’s complement $X - A$ is the union of a $r\omega$-open set and a $r\omega$-closed set.
(ii) Every $r\omega$-open (resp. $r\omega$-closed ) subset of $(X, \tau)$ is a $r\omega$-LC set.
(iii) The Complement of a $r\omega$-LC set need not be a $r\omega$-LC set.
(In Example 3.2 the set $\{b, c\}$ is $r\omega$-LC set, but complement of $\{b, c\}$ is $\{a, d\}$, which is not $r\omega$-LC set.)

3.4 Theorem: Every locally closed set is a $r\omega$-LC set but not conversely.
Proof: The proof follows from the two definitions [follows from Lemma 2.1] and fact that every closed (resp.open) set is $r\omega$-closed (rw-open).

3.5 Example: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ then $\{a, b\}$ is $r\omega$-LC set but not a locally closed set in $(X, \tau)$.
3.6 Theorem: Every $w$-lc set is a $r\omega$-LC set but not conversely.
Proof: The proof follows from the two definitions [follows from Lemma 2.1] and fact that every $w$-closed (resp.$w$-open) set is $r\omega$-closed (rw-open).

3.7 Example: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$ then $\{a, c\}$ is $r\omega$-LC set but not a $w$-locally closed set in $(X, \tau)$.
3.8 Theorem: Every $\theta$-lc set is a $r\omega$-LC set but not conversely.
Proof: The proof follows from the two definitions [follows from Lemma 2.1] and fact that every $\theta$-closed (resp.$\theta$-open) set is $r\omega$-closed (rw-open).

3.9 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then $\{c\}$ is $r\omega$-LC set but not a $\theta$-locally closed set in $(X, \tau)$.
3.10 Theorem: Every $I_\delta c$ set is a $r\omega$-LC set but not conversely.
Proof: The proof follows from the two definitions [follows from Lemma 2.1] and fact that every $I_\delta$-closed (resp.$I_\delta$-open) set is $r\omega$-closed (rw-open).

3.11 Example: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ then $\{a, b\}$ is $r\omega$-LC set but not a $I_\delta c$ set in $(X, \tau)$.
3.12 Theorem: Every $r\omega$-LC set is $rg$-lc set but not conversely.
Proof: The proof follows from the two definitions [follows from Lemma 2.1] and fact that every $r\omega$-closed (resp. $rw$-open) set is $rg$-closed (rg-open).

3.13 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then $\{a, d\}$ is $rg$-lc set but not $r\omega$-LC set in $(X, \tau)$.
3.14 Remark: $I_\delta gc$-sets and $r\omega$-LC sets are independent of each other as seen from the following example

3.15 Example: (i) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then $\{a, d\}$ is $I_\delta gc$-lc but not $r\omega$-LC set in $(X, \tau)$.
(ii) Let $X=\{a, b, c, d\}$ and $T=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then $\{c\}$ is RW-LC but not $\Theta$-lc set in $(X, \tau)$.

3.16 Remark: $\Theta$-lc sets and RW-LC sets are independent of each other as seen from the following example.

3.17 Example: i) Let $X=\{a, b, c, d\}$ and $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ then $\{b, d\}$ is $\Theta$-lc but not RW-LC set in $(X, \tau)$.

(ii) Let $X=\{a, b, c\}$ and $\tau=\{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ then $\{a, b\}$ is RW-LC but not $\Theta$-lc set in $(X, \tau)$.

3.18 Definition: A subset $A$ of $(X, \tau)$ is called a RW-LC$^\star$ set if there exists an open set $G$ and a closed $F$ of $(X, \tau)$ s.t. $A=G\cap F$.

3.19 Definition: A subset $B$ of $(X, \tau)$ is called a RW-LC$^{**}$ set if there exists an open set $G$ and a closed $F$ of $(X, \tau)$ s.t. $B=G\cap F$.

3.20 Theorem

1. Every locally closed set is a RW-LC$^\star$ set.
2. Every locally closed set is a RW-LC$^{**}$ set.
3. Every RW-LC$^\star$ set is RW-LC set.
4. Every RW-LC$^{**}$ set is RW-LC set.
5. Every RW-LC set is RW-LC$^\star$ set.
6. Every RW-LC$^{**}$ set is RW-LC$^{**}$ set.

Proof: The proof are obvious from the definitions and the relation between the sets.

However, the converses of the above results are not true as seen from the following examples.

3.21 Example: Let $X=\{a, b, c\}$ and $\tau=\{X, \emptyset, \{a\}\}$

(i) The set $\{b\}$ is RW-LC$^\star$ set but not a locally closed set in $(X, \tau)$.
(ii) The set $\{a, b\}$ is RW-LC$^{**}$ set but not a locally closed set in $(X, \tau)$.
(iii) The set $\{c\}$ is RW-LC$^\star$ set but not a $\omega$-lc$^\star$ set in $(X, \tau)$.
(iv) The set $\{c\}$ is RW-LC$^{**}$ set but not a $\omega$-lc$^{**}$ set in $(X, \tau)$.

3.22 Example: Let $X=\{a, b, c, d\}$ and $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$

(i) The set $\{a, b, d\}$ is RW-LC set but not a RW-LC$^\star$ set in $(X, \tau)$.
(ii) The set $\{a, d\}$ is $\omega$-lc$^\star$ set but not a RW-LC$^\star$ set in $(X, \tau)$.
(iii) The set $\{a, d\}$ is $\omega$-lc$^{**}$ set but not a RW-LC$^{**}$ set in $(X, \tau)$.
(iv) The set $\{b, d\}$ is $\omega$-lc set but not a RW-LC$^\star$ set in $(X, \tau)$.
(v) The set $\{b, d\}$ is $\omega$-lc set but not a RW-LC$^{**}$ set in $(X, \tau)$.
(vi) The set $\{b, d\}$ is $\omega$-lc$^\star$ set but not a RW-LC set in $(X, \tau)$.

3.23 Example: Let $X=\{a, b, c, d\}$ and $\tau=\{X, \emptyset, \{a\}, \{b\}, \{a, d\}, \{a, b, d\}\}$

(i) The set $\{b\}$ is RW-LC$^\star$ set but not a $\Theta$-lc$^\star$ set in $(X, \tau)$.
(ii) The set $\{d\}$ is RW-LC$^{**}$ set but not a $\Theta$-lc$^{**}$ set in $(X, \tau)$.

3.24 Example: Let $X=\{a, b, c\}$ and $\tau=\{X, \emptyset, \{a\}, \{a, c\}, \{b, c\}\}$

(i) The set $\{c\}$ is RW-LC$^\star$ set but not a $\Theta$-lc$^\star$ set in $(X, \tau)$.
(ii) The set $\{c\}$ is RW-LC$^{**}$ set but not a $\Theta$-lc$^{**}$ set in $(X, \tau)$.

3.25 Example: i) Let $X=\{a, b, c\}$ and $\tau=\{X, \emptyset, \{a, b, c\}\}$ then the set $\{c\}$ is RW-LC set but not a RW-LC$^{**}$ set in $(X, \tau)$

ii) Let $X=\{a, b, c\}$ and $\tau=\{X, \emptyset, \{a, b\}\}$ then the set $\{c\}$ is RW-LC set but not a $\alpha$-lc set in $(X, \tau)$.

3.26 Remark: RW-LC$^\star$ sets and RW-LC$^{**}$ sets are independent of each other as seen from the examples.
3.27 Example: (i) Let X={a,b,c,d} and τ={X,Φ, {a}, {b}, {a,b}, {a,b,c}} then set {a,b,d} is RW-LC** set but not a RW-LC* set in (X, τ).

(ii) Let X={a,b,c,d} and τ={X,Φ, {a}, {b}, {a,b}} then set {c} is RW-LC* set but not a RW-LC** set in (X, τ).

3.38 Remark: From the above discussion and known results we have the following implications in the diagram.

3.39 Theorem: If RWO(X, τ) = τ then

(i) RW-LC(X, τ) = LC (X, τ).
(ii) RW-LC(X, τ) = W-LC (X, τ).
(iii) RW-LC(X, τ) ⊆ GLC(X, τ).

Proof: (i) For any space (X, τ), W.k.t LC(X, τ) ⊆ RW-LC(X, τ). Since RWO(X, τ) = τ, that is every rw-open set is open and every rw-closed set is closed in (X, τ), RW-LC(X, τ) ⊆ LC (X, τ); hence RW-LC(X, τ) = LC (X, τ).

(ii) For any space (X, τ), LC(X, τ) ⊆ W-LC(X, τ)⊆ RW-LC(X, τ) From (i) it follows that RW-LC(X, τ) = W-LC(X, τ).

(iii) For any space (X, τ), LC(X, τ) ⊆ GLC (X, τ) from (i) RW-LC(X, τ) = LC (X, τ) and hence RW-LC(X, τ) ⊆ GLC (X, τ).

3.40 Theorem: If RWO(X, τ) = τ, then RW-LC*(X, τ) = RW-LC** (X, τ) = RW-LC(X, τ).

Proof: For any space (X, τ) LC(X, τ) ⊆ RW-LC*(X, τ) ⊆ RW-LC(X, τ) and LC (X, τ) ⊆ RW-LC**(X, τ) ⊆ RW-LC(X, τ) since RWO(X, τ) = τ. RW-LC(X, τ) =LC(X, τ) by theorem 3.39, it follows that LC(X, τ) =RW-LC*(X, τ) = RW-LC**(X, τ) = RW-LC(X, τ).

3.41 Remark: The converse of the theorem 3.40 need not be true in general as seen from the following example.

3.42 Example: Let X={a, b, c} with the topology τ= {X, Φ, {a}, {b}, {a,b}} then RW-LC*(X, τ) = RW-LC**(X, τ) = RW-LC(X, τ) = P(X). However RWO(X, τ)={X, Φ, {a}, {b}, {a,b}} ≠ τ.

3.43 Theorem: If GO(X, τ) = τ, then GLC(X, τ) ⊆ RW-LC(X, τ)

Proof: For any space (X, τ) w.k.t LC(X, τ) ⊆ GLC(X, τ) and LC (X, τ)⊆ RW-LC(X, τ)....(i) GO(X, τ) = T, that is every g-open set is open and every g-closed set is closed in (X, τ) and so GLC(X, τ) ⊆ LC(X, τ) that is GLC(X, τ) =LC(X, τ).....(ii) from (i) and (ii) we have GLC(X, τ) ⊆ RW-LC(X, τ).

3.44 Theorem: If RWC(X, τ) ⊆ LC(X, τ) then RW-LC(X, τ) = RW-LC*(X, τ)

Proof: Let RWC(X, τ) ⊆ LC(X, τ). For any space (X, τ), w.k.t RW-LC*(X, τ) ⊆ RW-LC(X, τ)....(i) Let A∈ RWC(X, τ), then A= U∈ F, where U is rw-open and F is a rw-closed in (X, τ). Now, F=RW-LC(X, τ) by hypothesis F is locally closed set in (X, τ), then F= G∪ E, where G is an open set and E is a closed set in (X, τ).

Now, A= U∈ F=∪ (G∪ E)= (∪ G)∪ E, where U∈ G is rw-open as the intersection of rw-open sets is rw-open and E is a closed set in (X, τ). It follows that A is RW-LC*(X, τ). That is A∈ RW-LC*(X, τ) and so, RWC(X, τ) ⊆ RW-LC*(X, τ)....(ii).

From (i) and (ii) we have RW-LC(X, τ) = RW-LC*(X, τ).

3.45 Remark: The converse of the theorem 3.44 need not be true in general as seen from the following example.

3.46 Example: Consider X= {a,b,c,d} and τ= {X, Φ, {a,b}, {c,d}} then RW-LC(X, τ) = RW-LC*(X, τ) = P(X). But RWC(X, τ) =P(X) and LC(X, τ) = {X, Φ, {a,b}, {c,d}} That is RWC(X, τ) ⊆ LC(X, τ).

3.47 Theorem: For a subset A of (X, τ) if A∈ RW-LC(X, τ) then A= U∈ (rw-cl(A)) for some open set U.
Proof: Let $A \in \text{RW-LC}(X, t)$ then there exist a rw-open $U$ and a rw-closed set $F$ s.t. $A = U \cap F$. Since $A \subseteq F$, $\text{rw-cl}(A) = \text{rw-cl}(F) = F$. Now $\bigcap (\text{rw-cl}(A)) \subseteq U \cap F = A$, that is $\bigcap (\text{rw-cl}(A)) = A$.

Conversely $A \subseteq U$ and $A \subseteq \text{rw-cl}(A)$ implies $A \subseteq U \cap (\text{rw-cl}(A))$ and therefore $A = U \cap (\text{rw-cl}(A))$ for some rw-open set $U$.

3.48 Remark: The converse of the theorem 3.47 need not be true in general as seen from the following example.

3.49 Example: Consider $X= \{a,b,c,d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ then

$\text{RWC}(X, t) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$

Take $A = \{b,d\}$, $\text{rw-cl}(A) = \{b,d\}$ now, $A = X \cap (\text{rw-cl}(A))$ for some rw-open set $X$ but $\{b,d\} \notin \text{RW-LC}(X, t)$.

3.50 Theorem: For a subset $A$ of $(X, t)$, the following are equivalent.

(i) $A \in \text{RW-LC}^*(X, t)$

(ii) $A = U \cap \text{cl}(A)$ for some rw-open set $U$.

(iii) $\text{cl}(A)$ is rw-closed.

(iv) $A \cup (\text{cl}(A))^c$ is rw-open.

Proof: (i) implies (ii) Let $A \in \text{RW-LC}^*(X, t)$ then there exists a rw-open set $U$ and a closed set $F$ s.t. $A = U \cap F$. Since $A \subseteq F$, $\text{cl}(A) \subseteq \text{cl}(F) = F$. Now $U \cap \text{cl}(A) \subseteq U \cap F = A$ that is $U \cap \text{cl}(A) = A$. Conversely $A \subseteq U$ and $A \subseteq \text{cl}(A)$ implies $A \subseteq U \cap \text{cl}(A)$ and therefore $A = U \cap \text{cl}(A)$ for some rw-open set $U$.

(ii) implies (i) Since $U$ is a rw-open set and $\text{cl}(A)$ is a closed set, $A = U \cap \text{cl}(A) \in \text{RW-LC}^*(X, t)$.

(iii) implies (iv) Let $F = \text{cl}(A)$, then $F$ is rw-closed by the assumption and $X - F = X - \text{cl}(A) = X \setminus \text{cl}(A) = A \cup (X - \text{cl}(A)) = A \cup (\text{cl}(A)) = A \cup (\text{cl}(A))^c$. But $X - F$ is open. This shows that $A \cup (\text{cl}(A))^c$ is rw-open.

(iv) implies (ii) Let $U = A \cup (\text{cl}(A))^c$ then $U$ is rw-open, this implies $X - U$ is rw-closed and $X - U = X - (A \cup (\text{cl}(A))^c) = \text{cl}(A) \cap (X - A) = \text{cl}(A)$ is rw-closed.

Therefore $A = U \cap (\text{cl}(A))$ for some rw-open set $U$.

(iii) implies (iv) Let $A = U \cap (\text{cl}(A))$ for some rw-open set $U$ then $A \in \text{RW-LC}^*(X, t)$. Now $A \cup (\text{cl}(A))^c = (U \cap (\text{cl}(A))) \cup (\text{cl}(A))^c = U \cap (\text{cl}(A))^c \cup \text{cl}(A) = U \cap X = X$ which is rw-open. Thus $A = (\text{cl}(A))^c$ is rw-open.

3.51 Theorem: For a subset $A$ of $(X, t)$ if $A \in \text{RW-LC}^{**}(X, t)$, then there exists an open set $U$ s.t. $A = U \cap \text{cl}(A)$.

Proof: Let $A \in \text{RW-LC}^{**}(X, t)$, then there exist an open set $U$ and a rw-closed set $F$ s.t. $A = U \cap F$. Since $A \subseteq U$ and $A \subseteq \text{rw-cl}(A)$ we have $A \subseteq \text{rw-cl}(A)$.

Conversely, Since $A \subseteq F$ and $\text{rw-cl}(A) \subseteq \text{rw-cl}(F) = F$, as $F$ is rw-closed. Thus $U \cap \text{rw-cl}(A) \subseteq U \cap F = A$. That is $U \cap \text{rw-cl}(A) \subseteq A$; hence $A = U \cap \text{rw-cl}(A)$. For some open set $U$.

3.52 Remark: The converse of the theorem 3.27 need not be true in general as seen from the following example.

3.53 Example: Let $X= \{a,b,c,d\}$ with the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ then $A = \{a,d\}$. Then $\text{rw-cl}(A) = \text{rw-cl}(\{a,d\}) = \{a,d\}$; also $A = X \cap \text{rw-cl}(A) = \{a,b,c,d\} \cap \{a,d\} = \{a,d\}$ for some open set $X$ but $\{a,d\} \notin \text{RW-LC}^{**}(X, t)$.

3.54 Theorem: For A and B in $(X, t)$ the following are true.

(i) If $A \in \text{RW-LC}^*(X, t)$ and $B \in \text{RW-LC}^*(X, t)$, then $A \cap B \in \text{RW-LC}^*(X, t)$.

(ii) If $A \in \text{RW-LC}^{**}(X, t)$ and B is open, then $A \cap B \in \text{RW-LC}^{**}(X, t)$.

(iii) If $A \in \text{RW-LC}(X, t)$ and B is open, then $A \cap B \in \text{RW-LC}(X, t)$.

(iv) If $A \in \text{RW-LC}^*(X, t)$ and B is rw-open, then $A \cap B \in \text{RW-LC}^*(X, t)$.

(v) If $A \in \text{RW-LC}^{**}(X, t)$ and B is closed, then $A \cap B \in \text{RW-LC}^{**}(X, t)$.

Proof: (i) Let $A, B \in \text{RW-LC}^*(X, t)$, it follows from theorem 3.---- that there exist rw-open sets $P$ and $Q$ s.t. $A = P \cap \text{cl}(A)$ and $B = Q \cap \text{cl}(B)$. Therefore $A \cap B = P \cap \text{cl}(A) \cap Q \cap \text{cl}(B) = P \cap Q \cap [\text{cl}(A) \cap \text{cl}(B)]$ where $P \cap Q$ is rw-open and $\text{cl}(A) \cap \text{cl}(B)$ is closed. This shows that $A \cap B \in \text{RW-LC}^*(X, t)$.

(ii) Let $A \in \text{RW-LC}^{**}(X, t)$ and B is open. Then there exist an open set $P$ and rw-closed set F s.t. $A = P \cap F$. Now, $A \cap B = P \cap F \cap B = (P \cap F) \cap F$, Where $(P \cap F)$ is open and F is rw-closed. This implies $A \cap B \in \text{RW-LC}^{**}(X, t)$. 

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(iii) Let $A \in \text{RW-LC}(X, \tau)$ and $B$ is rw-open then there exists a rw-open set $P$ and $Q$ rw-closed set $F$ s.t $A = P \cap F$. Now, $A \cap B = P \cap F \cap B = (P \cap F) \cap B$, Where $(P \cap F)$ is rw-open and $F$ is rw-closed. This shows that $A \cap B \in \text{RW-LC}(X, \tau)$.

(iv) Let $A \in \text{RW-LC}^*(X, \tau)$ and $B$ is rw-open then there exists a rw-open set $P$ and $Q$ rw-closed set $F$ s.t $A = P \cap F$. Now, $A \cap B = (P \cap F) \cap B = (P \cap B) \cap F$, Where $(P \cap B)$ is rw-open and $F$ is closed. This implies that $A \cap B \in \text{RW-LC}^*(X, \tau)$.

(v) $A \in \text{RW-LC}^*(X, \tau)$ and $B$ is closed. Then there exist an rw-open set $P$ and a closed set $F$ s.t $A = P \cap F$. Now, $A \cap B = (P \cap F) \cap B = P \cap (F \cap B)$, Where $(F \cap B)$ is closed and $P$ is rw-open. This implies $A \cap B \in \text{RW-LC}^*(X, \tau)$.

3.55 Definition: A topological space $(X, \tau)$ is called RW-submaximal if every dense set in it is RW-open.

3.56 Theorem: If $(X, \tau)$ is submaximal space then it is RW-submaximal space but converse need not be true in general.
Proof: Let $(X, \tau)$ be submaximal space and $A$ be a dense subset of $(X, \tau)$. Then $A$ is open. But every open set is rw-open and so $A$ is rw-open. Therefore $(X, \tau)$ is a RW-submaximal space.

3.57 Example: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then Topological space $(X, \tau)$ is RW-submaximal but set $A = \{a, b\}$ is dense in $(X, \tau)$ but not open therefore $(X, \tau)$ is not submaximal.

3.58 Theorem: A topological space $(X, \tau)$ RW-submaximal if and only if $P(X) = \text{RW-LC}^*(X, \tau)$.

Proof: 

Necessity: Let $A \subseteq P(X)$ and $U = A \cup (X-\text{cl}(A))$. Then it follows $\text{cl}(U) = \text{cl}(A \cup (X-\text{cl}(A))) = \text{cl}(A \cup (X-\text{cl}(A)) = X$. Since $(X, \tau)$ is RW-sub maximal, $U$ is rw-open, so $A \in \text{rwLC}^*(X, \tau)$ from the Theorem 3.47 Hence $P(X) = \text{rwLC}^*(X, \tau)$.

Sufficiency: Let $A$ be dense sub set of $(X, \tau)$.Then by assumption and Theorem 3.50 (iv) that $A \cup (X-\text{cl}(A)) = A$ holds, $A \in \text{rwLC}^*(X, \tau)$ and $A$ is rw-open. Hence $(X, \tau)$ is RW-sub maximal.

3.59 Theorem: If $(X, \tau)$ $T_{\text{rw}}$-space then $\text{RW-LC}(X, \tau) = \text{LC}(X, \tau)$.

Proof: Straight Forward.

3.60 Theorem: Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces.

i) If $A \in \text{RW-LC}(X, \tau)$ and $B \in \text{RW-LC}(Y, \sigma)$ then $A \times Y \in \text{RW-LC}(X \times Y, \tau \times \sigma)$.

ii) If $A \in \text{RW-LC}^*(X, \tau)$ and $B \in \text{RW-LC}^*(Y, \sigma)$ then $A \times B \in \text{RW-LC}^*(X \times Y, \tau \times \sigma)$.

iii) If $A \in \text{RW-LC}**(X, \tau)$ and $B \in \text{RW-LC}**(Y, \sigma)$ then $A \times B \in \text{RW-LC}**(X \times Y, \tau \times \sigma)$.

Proof: i) If $A \in \text{RW-LC}(X, \tau)$ and $B \in \text{RW-LC}(Y, \sigma)$. Then there exist rw-open sets $U$ and $V$ of $(X, \tau)$ and $(Y, \sigma)$ and rw-closed sets $G$ and $F$ of $X$ and $Y$ respectively such that $A = U \cap G$ and $B = V \cap F$. Then $A \times B = (U \times V) \cap (G \times F)$ holds. Hence $A \times B \in \text{RW-LC}(X \times Y, \tau \times \sigma)$.

ii) and iii) Similarly the follow from the definition.

4. References


4. Arockiarani I, Balachandran K. On $\theta$-g locally closed set (pre print).


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