Product of finitely permutable groups

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Abstract
In this paper we show that if $A_1, A_2, \ldots, A_n$ are finitely many pairwise permutable abelian min-by-max subgroups of the group $G$ such that $G=A_1 \cdots A_n$, then $J(G)=J(A_1) \cdots J(A_n)$, which is $J(G)$ finitely residual of a group $G$.

Keywords: Minimal condition, maximal condition, finite residual group

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1. Introduction
In 1940 G. Zappa (See [21]) and in 1950 J. Szip (See [20]) studied about products of groups concerned finite groups. In 1961 O.H. Kegel (See [8]) and in 1958 H. Wielandt (See [10]) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups.
In 1955 N. Itô (See [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (See [18]) and L. Redei (1950) (See [19]) considered products of cyclic groups, and around 1965 O.H. Kegel (See [23, 24]) looked at linear and locally finite factorized groups.
In 1968 N.F. Sesekin (See [16]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition. He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [17, 1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product $G$ of two subgroups $A$ and $B$ satisfying a certain finiteness condition $\star$, when does $G$ have the same finiteness condition $\star$? (See [17])
For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (See [1-4, 6]), N.S. Chernikov (See [5]), S. Franciosi, F. de Giovanni (See [3, 6, 25-29]), O.H. Kegel (See [8]), J.C. Lennox (See [12]), D.J.S. Robinson (See [9, 14]), J.E. Roseblade (See [13]), Y.P. Sysak (See [30-33]), J.S. Wilson (See [34]), and D.I. Zaitsev (See [11, 15]).
Now, in this paper, we study the residual finite group and min-by-max subgroups of the group $G$ and its relations, and the end we prove that if $A_1, A_2, \ldots, A_n$, are finitely many pairwise permutable abelian min-by-max subgroups of the group $G$ such that $G$ is the products of $A_1, \ldots, A_n$, then $G$ is soluble min-by-max-group and $J(G)$ is products of $J(A_1), \ldots, J(A_n)$, i.e. $J(G) = J(A_1) \cdots J(A_n)$. For do this, in chapter 2 we express the elementary lemmas and Theorems and in chapter 3 we prove the main Theorem.

2. Preliminaries (Elementary properties and Theorems.)
In this chapter we express the elementary Lemma and definitions whose used in prove the Main Theorem in chapter.
2.1 Theorem (See [11, 12]): If the soluble-by-finite group $G=AB$ is the product of two polycyclic-by-finite subgroups $A$ and $B$, then $G$ is polycyclic-by-finite.

Proof: Assume that $G$ is not polycyclic-by-finite. Then $G$ contains an abelian normal section $U/V$ which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of $U/V$ in $G/V$ is also a counterexample. Hence we may suppose that $G$ has a triple factorization $G=AB=AK= BK$, where $K$ is an abelian normal subgroup of $G$ which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group $G$ satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup $M$ which is maximal with respect to the condition that $G/M$ is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of $G$ is polycyclic-by-finite.

2.2 Theorem (See [15]): Let the soluble group $G=AB$ be the product of two subgroups $A$ and $B$ with finite abelian section rank. If at least one of the factors $A$ and $B$ has an ascending normal series with central or periodic factors, then $G$ also has finite abelian section rank.

Proof: See ([4], Theorem 4.6.10).

2.3 Theorem (See [6]): Let the group $G=AB=AK= BK$ be the product of three nilpotent subgroups $A$, $B$, and $K$, where $K$ is normal in $G$. If $K$ is minimax, then $G$ is nilpotent.

Proof: See ([4], Theorem 6.3.4).

2.4 Theorem (See [6]): Let the group $G=AB=AK= BK$ be the product of two subgroups $A$ and $B$ and a minimax normal subgroup $K$ of $G$.

(i) If $A$, $B$, and $K$ are locally nilpotent, then $G$ is locally nilpotent.

(ii) If $A$, $B$, and $K$ are hypercentral, then $G$ is hypercentral.

Proof: See ([4], Theorem 6.3.7).

2.5 Lemma: Let the group $G=AB$ be the product of two abelian subgroups $A$ and $B$ such that $A_G=B_G=1$. Then the following hold.

(i) $A \cap B = Z(G) = 1$.

(ii) $A \cap C_G(G') = B \cap C_G(G') = 1$, and in particular $A \cap G' = B \cap G' = 1$.

(iii) The factorizer $X = X(G')$ of $G'$ does not have non-trivial normal subgroups which are contained in $A$ or $B$, so that in particular $X(\langle x \rangle) = 1$.

(iv) The FC-centre of $G$ is trivial.

Proof: (i) They Lemma 2.7 we have that $Z(G) = (A \cap Z(G))(B \cap Z(G)); A_G=B_G=1$.

Hence $Z(G)=1$. Moreover, $A \cap B$ in contained in $Z(G)$ and so is also trivial.

(ii) This follows from the first part of the proof of Lemma 2.9.

(iii) Let $N$ be a normal subgroup of $X$ contained in $A$. Then $G'$ normalizes $N$, so that by (ii) $[N, G'] = N \cap G = A \cap G' = 1$.

Therefore $N$ is contained in $A \cap C_G(G') = 1$.

(iv) Let be an element of $A \cap F$, where $F$ is the FC-centre of $G$. Since $G'$ is abelian by Theorem 2.5, the mapping $\varphi : x \mapsto [x, a]$ is a $G$ epimorphism from $G'$ onto $[G', a]$. Hence $C_G(a) = \ker \varphi$ is a normal subgroup of $G$.

Proof: Assume that $G$ is not polycyclic-by-finite. Then $G$ contains an abelian normal section $U/V$ which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of $U/V$ in $G/V$ is also a counterexample. Hence we may suppose that $G$ has a triple factorization $G=AB=AK= BK$, where $K$ is an abelian normal subgroup of $G$ which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group $G$ satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup $M$ which is maximal with respect to the condition that $G/M$ is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of $G$ is polycyclic-by-finite.

2.6 Theorem: (See [22]): Let the group $G=AB \neq 1$ be the product of two abelian subgroups $A$ and $B$, at least one of which has finite section rank. Then there exists a non-trivial normal subgroup of $G$ contained in $A$ or $B$.

Proof: Assume that $A_G = B_G = 1$, so that $A \cap G' = B \cap G' = 1$ by Lemma 2.15(ii). The factorizer $X = X(G')$ has the triple factorization $X = (A \cap B)G'(B \cap AG') = (A \cap BG')G' = (B \cap AG')G'$, and its centre is trivial by Lemma 2.15(iii). The subgroups $A \cap BG'$ and $B \cap AG'$ are isomorphic, and hence both have finite section rank. By Theorem 2.12 the metabelian group $X$ has finite abelian section rank, and hence is hypercentral by Theorem 2.14. In particular $Z(X) \neq 1$, a contradiction.

2.17 Theorem: (See [15]): Let the group $G=A_1...A_n$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,...,A_n$. Then $G$ is a soluble minimax group.

Proof: Assume that the theorem is false, and let $G=A_1...A_n$ be a counterexample for which the sum $\sum_{i=1}^n m(A_i)$ is minimal. Suppose that there are indices $i<j$ such that $D = A_i \cap A_j$ is infinite. Then $D = D_{A_i} = D_{A_1...A_j}$ is also a soluble minimax group. It follows that $D_{A_i}$ is a soluble minimax group. On the other hand, the factor group $G = G/D_{A_i}$ is also a soluble minimax group since $m(A_i) < m(A_j)$. This contradiction shows that $A_i \cap A_j$ is finite if $i \neq j$.

Let $J_i$ be the finite residual of $A_i$ for every $i=1,...,t$. It follows from lemma 2.15 that $J_i J_i$ is the finite residual of the soluble minimax group $A_i A_i$, so that it is abelian. Hence $L = \langle J_1,...,J_t \rangle$ is an abelian group satisfying the minimal condition. As $[A_i, J_j] \leq J_i \leq L$, the subgroup $L$ is normal in $G$. Assume that $J_i \neq 1$ for some $i$. Then $m(A_i, L/L) < m(A_i)$, and so $G/L$ is a soluble minimax group. This contradiction proves that $J_i = 1$ for each $i$.
particular the maximum periodic normal subgroup $E$ of $A_1A_2$ is finite. If $A_1A_2=E$, then the soluble minimax group by Corollary 2.10 Thus $E$ is properly contained in $A_1A_2$, and by Theorem 2.16 we may suppose that $A_iE/E$ contains a non-trivial normal subgroup $N/E$ of $A_1A_2E=(A_1E/E)(A_2E/E)$. As $A_1A_2E$ has no finite-non-trivial normal subgroups, $N/E$ must be infinite. Moreover, the index $|N:N_1|\leq|A_1N:A_1|\leq|A_1:E:A_1|$ is finite. If $M$ is the core of $N\cap A_1$ in $A_1A_2$, then $N/M$ has finite exponent and hence is finite. Therefore $M$ is an infinite normal subgroup of $A_1A_2$ contained in $A_i$. Since $M^o=M^{A_1-A_i}\leq A_2\cdots A_i$, it follows that $M^o$ is a soluble minimax group. As above, $G/M^o$ is also a soluble minimax group since $m(A_1M^o/M^o)<m(A_1)$ This contradiction proves the theorem.

3. Main Result: In this chapter by used the Lemmas and Theorems of chapter 2, we prove the Basic theorem of this paper as follows.

3.1. Main Theorem: Let the group $G=A_1\cdots A_n$ be the product of finitely many pairwise permutable abelian min-by-max subgroups $A_1,\cdots A_n$. Then $G$ is a soluble min-by-max group and $J(G)=J(A_1)\cdots J(A_t)$.

Proof: It follows from Theorem 2.17 that $G$ is a soluble minimax group, and hence $J(G)$ is abelian. Put $J_i=J(A_i)$ for each $i=1,\ldots,t$. Then $L=J_1\cdots J_i$ is contained in $J$. Let $I$ be the finite residual of $A_iA_i$. The factorizer $X=X(I)$ of $I$ in $A_iA_i$ has the triple factorization $X=A_i^*A_i^*=A_iI=A_iI$, where $A_i^*=A_i\cap A_iI$ and $A_i^*=A_i\cap A_iI$. It follows that $J_i$ and $J_j$ are contained in $Z(X)$, and the factor group $X/J_iJ_j$ is polycyclic by Theorem 2.11. Therefore $J_iJ_j$ is the finite residual of $X$ and so $J_iJ_j=I$. Thus $[A_i,J_i]\leq J_iJ_i\leq L_i$, and hence $L$ is normal in $G$. The factor group $A_i/L_i$ is polycyclic for every $i\leq t$ and hence also $G/L=(A_1/L_1)\cdots (A_t/L_t)$ is polycyclic by Theorem 2.10. This proves that $G$ is a min-by-max group and $J=L=J_1\cdots J_t$.

4. Reference


