Hyers – Ulam stability of linear difference equations of first order

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Abstract
We prove the Hyers – Ulam stability of linear difference equations of first order of the form
\[ \phi(t) \Delta y(t) = y(t) \]  

Keywords: Hyers – Ulam stability, difference equation, first order.

1. Introduction
The theory of difference equations and their applications have been receiving intensive attention. See, for example [1-3] and the references cited therein.

In this paper we consider the first order difference equation of the form
\[ \phi(t) \Delta y(t) = y(t), \quad t \in I \]  
where, \( I = \mathbb{N}(a) = \{a, a+1, a+2, \ldots\} \), (a is a fixed nonnegative integer), \( \Delta \) is the forward difference operator defined by \( \Delta y(t) = y(t+1) - y(t) \). Assume further that \( \phi: I \rightarrow \mathbb{R} \) is a given function. By a solution of equation (1) we mean a sequence \( \{z(t)\} \) which is defined for \( t \in I \), and satisfies equation (1). S.M Jung the author [4] who investigated the Hyers –Ulam stability of linear differential equation of first order
\[ \phi(t) \sigma(t) = y(t), \]  
which was the improvement of the papers [5, 6]. Indeed they dealt with the Hyers Ulam stability of differential equation \( \sigma(t) = \theta(t) \), while Alsina and Ger investigated the differential equation \( \sigma(t) = y(t) \).

The aim of this paper is to investigate the Hyers –ulam stability of the linear difference equation of first order (1). More precisely, we prove that if \( \phi(t) > 0 \) or \( \phi(t) \leq -1 \) hold for all \( t \in I \) and further if the function \( y(t) \) satisfies
\[ |\phi(t) \Delta y(t) - y(t)| < \epsilon \quad \forall \ t \in I \]  
then there exists a real number \( c \) such that
\[ |y(t) - c \sum_{s=a}^{t-1} \theta(s)| \leq \epsilon \quad \forall \ t \in I. \]

2. Preliminaries
Following an idea of Scou – Mo Jung [4] we prove the following lemma.

Lemma 2.1 Assume that a function \( z: I \rightarrow \mathbb{R} \) is given, The inequality \( z(t) \leq \theta(t) \Delta z(t) \) is true for \( t \in I \), if and only if there exists a function \( \alpha: I \rightarrow \mathbb{R} \) such that
\[ \Delta z(t) = \alpha(t) \sum_{s=a}^{t-1} \theta(s) \quad \forall \ t \in I. \]

Proof: Assume the inequality \( z(t) \leq \theta(t) \Delta z(t) \) holds true for all \( t \in I \). Let us define function \( \alpha: I \rightarrow \mathbb{R} \) such that
\[ \alpha(t) = z(t) \sum_{s=a}^{t-1} \theta(s) \]  
then,
\[ \Delta \alpha(t) = \Delta \left( \sum_{s=a}^{t-1} \frac{\theta(s)}{1+\theta(s)} \right) z(t) + \sum_{s=a}^{t-1} \frac{\theta(s)}{1+\theta(s)} \Delta z(t) \]
\[ = \prod_{s=a}^{t-1} \frac{\theta(t)}{1+\theta(s)} [\Delta z(t)] + z(t+1) \frac{\theta(t)}{1+\theta(t)} \]
Since by hypothesis \( \alpha(t) \Delta z(t) \geq z(t) \)
\[ \alpha(t) \Delta z(t) \geq z(t) \prod_{s=a}^{t-1} \frac{\phi(s)}{\phi(s)} + z(t+1) \prod_{s=a}^{t} \frac{\phi(s)}{\phi(s)} > 0 \]
and
\[ \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} = z(t) \prod_{s=a}^{t-1} \frac{\phi(s)}{\phi(s)} \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} = z(t). \]

Conversely, assume that there exists a function \( \alpha: I \to R \) such that \( \alpha(t) \Delta z(t) \geq z(t) \) for each \( t \in I \). Let us define a function \( z: I \to R \) by,
\[ z(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)}, \]
then,
\[ \Delta z(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} + \alpha(t+1) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]
Therefore, \( \alpha(t) \Delta z(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} + \alpha(t+1) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \).

Since by hypothesis \( \alpha(t) \Delta z(t) \geq 0 \), we have
\[ \alpha(t) \Delta z(t) \geq \alpha(t+1) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \geq z(t), \forall t \in I \]
which proves lemma.

**Lemma 2.2** Assume that a function \( z: I \to R \) is given, The inequality \( z(t) \geq \alpha(t) \Delta z(t) \) holds true for any \( t \in I \), if and only if there exists a function \( \beta: I \to R \) such that \( \beta(t) \Delta z(t) < 0 \) and \( z(t) = \beta(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)}, \forall t \in I \).

**Proof:** Assume the inequality \( z(t) \geq \alpha(t) \Delta z(t) \) holds for all \( t \in I \). Let us define \( \beta: I \to R \) to be \( \beta(t) = \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \), then,
\[ \Delta \beta(t) = \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \Delta z(t) + \alpha(t+1) \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]
\[ \phi(t) \Delta \beta(t) = \phi(t) \Delta z(t) + \alpha(t+1) \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]
Since \( \phi(t) \Delta \beta(t) \leq z(t) \)
\[ \phi(t) \Delta \beta(t) \leq z(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} - z(t+1) \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]
and
\[ \beta(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} = z(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} = z(t). \]

Conversely, assume that \( \Delta \beta(t) \Delta z(t) \leq 0 \) for each \( t \in I \) and let us define a function \( z: I \to R \) by \( z(t) = \beta(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \), then,
\[ \phi(t) \Delta z(t) = \phi(t) \Delta \beta(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} + \beta(t+1) \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]

Since \( \phi(t) \Delta \beta(t) \leq 0 \),
\[ \phi(t) \Delta z(t) \leq \beta(t+1) \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]
Therefore \( \phi(t) \Delta z(t) \leq z(t) \). Hence the proof.

**Theorem 2.3** Given an \( \epsilon > 0 \), a function \( y: I \to R \) is a solution of the following inequality
\[ \left| \phi(t) \Delta y(t) \right| \leq \epsilon \]
if and only if there exists a function \( \alpha: I \to R \) such that
\[ y(t) = c + \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]
and
\[ 0 \leq \Delta \alpha(t) \leq 2 \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)}, \forall t \in I. \]

**Proof:** First we assume that \( y: I \to R \) is a solution of the inequality (2) then \( y(t) - c \leq \phi(t) \Delta y(t) \leq y(t) + c \)
for each \( t \in I \). Define \( z(t) = y(t) - c \) then \( \Delta z(t) = \Delta y(t) \), the inequality on the L.H.S of (5), becomes \( z(t) \leq \phi(t) \Delta z(t) \) holds for every \( t \in I \). According to lemma 1.1 there exists a function \( \alpha: I \to R \) such that
\[ y(t) = c + \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]
for all \( t \in I. \)

where \( \alpha \) additionally satisfies the condition,
\[ \Delta \alpha(t) \geq 0, \forall t \in I \]

Analogously define \( z(t) = y(t) + c \), the inequality on the R.H.S of (5) implies that \( z(t) \geq \phi(t) \Delta y(t) \geq \phi(t) \Delta z(t) \) holds for any \( t \in I \).

According to lemma 1.2 there exists a function \( \beta: I \to R \) such that \( y(t) + c = z(t) \)
and \( \Delta \beta(t) \leq 0, \forall t \in I \).

From (6),
\[ \Delta y(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]

From (8), \( \Delta y(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \)

Also, from (8) and (6)
\[ \beta(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} = 2 c + \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]

substituting in (11)
\[ \Delta y(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]

from (7) and (9) we get,
\[ 0 \leq \Delta \alpha(t) \phi(t) \leq 2 c \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]

Conversely, assume that \( y: I \to R \) is given by (3),
\[ y(t) = c + \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]
and a function \( \alpha: I \to R \) satisfies
\[ 0 \leq \Delta \alpha(t) \phi(t) \leq 2 c \prod_{s=a}^{t} \frac{1+\phi(s)}{\phi(s)} \]
then from (10),
\[ \phi(t) \Delta y(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]
Using (3) and (4)
\[ \phi(t) \Delta y(t) = \alpha(t) \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \]

By (4) and the last equation, we conclude that
\[ \left| \phi(t) \Delta y(t) \right| \leq \epsilon \]
Which proves the theorem.

3. **Hyers – Ulam Stability of difference equation (1)**
In the following theorem, we prove the Hyers-Ulam stability of the difference equation (1).

**Theorem 3.1** If either \( \phi(t) > 0 \) holds for all \( t \in I \), or \( \phi(t) < 0 \) holds for all \( t \in I \), and if a function \( y: I \to R \) satisfies the inequality (2) then there exists a real number \( \epsilon \) such that
\[ \left| y(t) - c \right| \prod_{s=a}^{t-1} \frac{1+\phi(s)}{\phi(s)} \leq \epsilon \]
**Proof:** First we assume that $\emptyset(t) > 0$ holds for all $t \in I$ and a function $y: I \to R$ satisfies the inequality (2) for all $t \in I$. Let $\alpha: I \to R$ is a function such that (3) and (4) hold, using Theorem 2.3

$$y(t) = \epsilon + \alpha(t) \prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)}$$

and

$$0 \leq \Delta \alpha(t)\emptyset(t) \leq 2\epsilon \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1 + \emptyset(s)}$$

Define $c = \lim_{t \to a} \alpha(t)$. Also we use $\prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)} = 1$.

Now we can divide the above inequality by $-(\emptyset(t))$ we get,

$$0 \geq -\Delta \alpha(t) \geq \frac{-2\epsilon}{(\emptyset(t))} \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1 + \emptyset(s)}$$

Since $1 + \emptyset(t) > \emptyset(t)$

Taking the summation on both sides, we get

$$0 \geq -\sum \Delta \alpha(t) \geq 2\epsilon \sum \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1 + \emptyset(s)}$$

$$0 \geq -\alpha(t) + c \geq 2\epsilon \left(\prod_{s=a}^{t-1} \frac{\emptyset(s)}{1 + \emptyset(s)}\right)$$

$$-\epsilon \geq -\alpha(t) \prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)} + c \prod_{s=a}^{t-1} \frac{1 - \emptyset(s)}{\emptyset(s)} \geq \epsilon \geq c$$

$$-\epsilon \leq \alpha(t) \prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)} + c \prod_{s=a}^{t-1} \frac{1 - \emptyset(s)}{\emptyset(s)} \leq \epsilon$$

$$-\epsilon \leq y(t) - c \prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)} \leq \epsilon$$

$$\left|y(t) - c \prod_{s=a}^{t-1} \frac{1 + \emptyset(s)}{\emptyset(s)}\right| \leq \epsilon$$

for any $t \in I$, which proves (12).

If we assume that $1 + \emptyset(t) < 0$ holds true for all $t \in I$, then the proof is similar to the above procedure, hence we omit it.

**4. Remark:** Here, we notice that $y(a) \prod_{s=a}^{t-1} \emptyset(s)$ is the general solution of the difference equation $\emptyset(t)\Delta y(t) = y(t)$, where $\emptyset(s) = \frac{1 + \emptyset(s)}{\emptyset(s)}$.

**5. References**