An extension of a theorem of Ito on conjugacy class sizes

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Abstract
Let G be a finite group and let $G^*$ be the set of elements of prime power order of G. In this paper we show that, if no conjugacy class size of $G^*$ is divisible by the product pq, then G is p-nilpotent with abelian Sylow p-subgroup or G is q-nilpotent with abelian Sylow q-subgroup, where p, q are distinct primes.

Keywords: Finite group, conjugacy class sizes, elements of prime power orders

1. Introduction
A well-established research area in finite group theory consists in exploring the relationship between the structure of a group G and certain sets of positive integers, which are naturally associated to G. One of those sets, denoted by $\text{cs}(G)$, is the set of conjugacy class sizes of the elements of G.

A classical remark concerning the influence of $\text{cs}(G)$ on the group structure of G is the following: if p is a prime number which does not divide any element of $\text{cs}(G)$, then G has a central Sylow p-subgroup (see [1, Theorem 33.4]). In [2], Ito proves the following well-known theorem: if no conjugacy class size of G is divisible by the product pq, then G is p-nilpotent with abelian Sylow p-subgroup or G is q-nilpotent with abelian Sylow q-subgroup, where p, q are distinct primes (see [2, Proposition 5.1]). In view of that, one can ask whether particular subsets of $\text{cs}(G)$ still encode nontrivial information on the structure of G. For instance, $\text{cs} \left( G^* \right)$, which is the set of conjugacy class sizes of the elements of prime power order of G. In [3] Kong and Guo obtain a complete extension of the above former result: if p is a prime number which does not divide any element of $\text{cs}(G^*)$, then G has a central Sylow p-subgroup (see [3, Lemma 2.4]).

In this paper, we will continue to focus our attention on $\text{cs}(G^*)$ and obtain a complete extension of above result of Ito. Our main result is the following:

Theorem A. Let G be a group and p, q distinct primes. If no conjugacy class size of $G^*$ is divisible by the product pq, then G is p-nilpotent with abelian Sylow p-subgroup or G is q-nilpotent with abelian Sylow q-subgroup.

2. Proof of the Main Theorem

Proof of Theorem A. Let P and Q be a Sylow p-subgroup and a Sylow q-subgroup of G, respectively. Now we write $M = C_G(P)$ and $N = C_G(Q)$. By assumption every element of prime power order of G centralizes some conjugate of either P or Q. Thus every element of G centralizes some conjugate of either P or Q. It follows that $G = \bigcup_{x \in G} M^x \cup \bigcup_{y \in G} N^y$.

We can assume that M and N are proper subgroups of G, as otherwise we are done. It follows that $|G : M| + |G : N| = |M| + |N|$, where the inequality is strict, since the identity of G is counted more than once. So $1 < \frac{1}{|N_G(M) : M|} \cdot \frac{1}{|N_G(N) : N|}$ and hence, say,
\( M = N_G(M) \). Since \( M = C_G(P) \leq N_G(P) \leq N_G(M) \), it follows that \( C_G(P) = N_G(P) \). Thus P is abelian and, by criterion of Burnside, G is p-nilpotent.

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References