Regular mildly generalized closed and regular mildly generalized open sets in bitopological spaces

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Abstract
The aim of this paper is to introduced the concept of \((\tau_i, \tau_j)\)-Regular Mildly Generalized (briefly \((\tau_i, \tau_j)\)-RMG) closed and \((\tau_i, \tau_j)\)-Regular Mildly Generalized (briefly \((\tau_i, \tau_j)\)-RMG) open sets and study their basic properties in bitopological spaces. Further we define and study new neighborhood namely \((\tau_i, \tau_j)\)-Regular Mildly Generalized (briefly \((\tau_i, \tau_j)\)-RMG-interior) interior and discuss some of their properties in bitopological spaces. Also give some characterizations and applications of it.

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1. Introduction
A triple\((X, \tau_1, \tau_2)\) where \(X\) is a non-empty set and \(\tau_1, \tau_2\) are topologies on \(X\) is called bitopological space. J. C. Kelly [3] initiated the systematic study of such spaces in 1963. In 1986, T. Fukutake [3] introduced the concept of generalized closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces. M. A. Jalic [4], N. Nagaveni [6], I. Arockiarani [1] and M. Sheik John [11] introduced \((i, j)\)-pre-open, \((i, j)\)-wg-closed, \((i, j)\)-rg-closed and \((i, j)\)-g' -closed sets in bitopological spaces. Also R. S. Wali and Nirani Laxmi[12, 13] introduce and studied the properties of RMG-closed and RMG-open sets in topological space. The purpose of this paper to introduced and investigate the concept of \((\tau_i, \tau_j)\)-RMG-closed and \((\tau_i, \tau_j)\)-RMG-open sets which are introduced in a bitopological space by analogy with RMG-closed and RMG-open sets in bitopological space.

2. Preliminaries
Throughout this paper the space \(X\) always means \((X, \tau_1, \tau_2)\) bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let \(A\) be a subset of a space \(X\). \(\tau_i\)-\text{int}(A), \(\tau_i\)-\text{cl}(A) we shall denote the interior and closure of \(A\subset X\) with respect to the topology \(\tau_i\) for \(i=1, 2\). By \((i, j)\) means pair of topologies \((\tau_i, \tau_j)\) and \(X-A\) or \(A'\) denotes the complement of \(A\) in \(X\).

Now we recall the following known definitions and results that are used in our work;

**Definition 2.1:** A subset \(A\) of a topological space \(X\) is called
(i) Pre-open [7], if \(A\subseteq \text{int}(\text{cl}(A))\) and pre-closed if \(\text{cl}(\text{int}(A))\subseteq A\).

**Definition 2.2:** A subset \(A\) of a topological space \(X\) is called
(i) Generalized closed (briefly g-closed) [6] if \(\text{cl}(A)\subseteq U\) whenever \(A\subseteq U\) and \(U\) is open in \(X\).
(ii) Weakly generalized closed (briefly wg-closed) [8] if \(\text{cl}(\text{int}(A))\subseteq U\) whenever \(A\subseteq U\) and \(U\) is open in \(X\).
(iii) Strongly generalized closed (briefly g*–closed) \(^{[11]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is g-open in \(X\).
(iv) Mildly generalized closed (briefly mildly g-closed) \(^{[10]}\) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is g-open in \(X\).
(v) Regular weakly closed (briefly rw-closed) \(^{[2]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular semi open in \(X\).
(vi) Regular generalized closed (briefly rg-closed) \(^{[9]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open set in \(X\).
(vii) Regular Mildly Generalized closed(briefly RMG-closed) \(^{[12]}\) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is rg-open in \(X\).

The complements of above all closed sets are their respective open sets in the same topological space \(X\).

**Definition 2.3:** Let \(i, j \in \{1, 2\}\) be fixed integers. A subset \(A\) of a bitopological space \((X, \tau_i, \tau_j)\) is said to
(i) \((\tau_i, \tau_j)\)-pre-open \([4]\) if \(A \subseteq \tau_i - \text{int}(\tau_j \cap \text{cl}(A))\)
The complement of \((\tau_i, \tau_j)\)-pre-open set is called \((\tau_i, \tau_j)\)-pre-closed set.

**Definition 2.4:** Let \(i, j \in \{1, 2\}\) be fixed integers. A subset \(A\) of a bitopological space \((X, \tau_i, \tau_j)\) is called
(i) \((i, j)\)-g–closed \(^{[3]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(\tau_i\).
(ii) \((i, j)\)-wg–closed \(^{[8]}\) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(\tau_i\).
(iii) \((i, j)\)-g*–closed \(^{[11]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(\tau_i\).
(iv) \((i, j)\)-rw–closed \(^{[3]}\) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular semi open in \(\tau_i\).
(v) \((i, j)\)-rg–closed \(^{[1]}\) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open set in \(\tau_i\).

The complements of above all closed sets are their respective open sets in the same bitopological space \(X\).

3.\((\tau_i, \tau_j)\)-Rmg-Closed Sets and Some of Their Properties.

**Definition 3.1:** Let \((i, j) \in \{1, 2\}\) be fixed integers. In a bitopological space \((X, \tau_i, \tau_2)\), a subset \(A \subseteq X\) is said to be \((i, j)\)-Regular Mildly Generalized closed (briefly, \((i, j)\)-Rmg-closed) set if \(\text{cl}(\text{int}(A)) \subseteq G\), whenever \(A \subseteq G\) and \(G \subseteq \text{RGO}(X, \tau_i)\).
We denote the family of all \((i, j)\)-Rmg-closed sets in a bitopological space \((X, \tau_i, \tau_2)\) by \(\text{D}_{\text{Rmg}}(\tau_i, \tau_2)\) or \(\text{D}_{\text{Rmg}}(i, j)\).

**Remark 3.2:** By setting \(\tau_1 = \tau_2\) in Definition 3.1, \((i, j)\)-Rmg closed set reduces to an Rmg-closed set in \(X\).
First we prove that the class of \((i, j)\)-Rmg-closed sets properly lies between the class of \((i, j)\)-pre-closed sets and the class of \((i, j)\)
mildly-g-closed sets.

**Theorem 3.3:** If \(A\) is a \((i, j)\)-pre-closed subset of \((X, \tau_i, \tau_2)\), then \(A\) is \((i, j)\)-Rmg-closed, but converse is not true.

**Proof:** Let \(A\) be a \((i, j)\)-pre-closed subset of \((X, \tau_i, \tau_2)\). Let \(U\) be rg-open in \((X, \tau_i)\) such that \(A \subseteq G\). Since \(A\) is \((i, j)\)-pre-closed subset of \((X, \tau_i, \tau_2)\) that is \(\text{cl}(\text{int}(A)) \subseteq \text{cl}(\text{int}(A)) \subseteq G\), we have \(\text{cl}(\text{int}(A)) \subseteq \text{cl}(\text{int}(A)) \subseteq G\). Therefore \(A\) is \((i, j)\)-Rmg-closed.

**Example 3.4:** Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}\). Then the subset \{a, c\} is \((1, 2)\)-Rmg closed set, but not \((1, 2)\)-pre-closed set in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 3.5:** If \(A\) be a \((i, j)\)-Rmg-closed subset of \((X, \tau_i, \tau_2)\), then \(A\) is \((i, j)\)-mildly-g-closed, but converse is not true.

**Proof:** Let \(A\) be a \((i, j)\)-Rmg-closed subset of \((X, \tau_i, \tau_2)\). Let \(G \subseteq \text{RGO}(X, \tau_i)\) be such that \(A \subseteq G\). Since \(G \subseteq \text{RGO}(X, \tau_i)\), we have \(G \subseteq \text{RGO}(X, \tau_i)\). Then by hypothesis, \(\text{cl}(\text{int}(A)) \subseteq G\). Therefore \(A\) is \((i, j)\)-mildly-g-closed.

**Example 3.6:** Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}\). Then the subsets \{a, b\} and \{a, b, d\} are \((1, 2)\)-mildly-g-closed sets, but not \((1, 2)\)-Rmg-closed sets in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 3.7:** If \(A\) is \(\tau_j\)-closed subset of bitopological space \((X, \tau_1, \tau_2)\), then the set \(A\) is \((i, j)\)-Rmg-closed, but converse is not true.

**Proof:** Let \(\text{RGO}(X, \tau_i)\) be such that \(A \subseteq G\). Then by hypothesis, \(\text{cl}(\text{int}(A)) \subseteq \text{cl}(A) = A\), which implies \(\text{cl}(\text{int}(A)) \subseteq G\). Therefore \(A\) is Rmg-closed.

**Example 3.8:** Let \(X = \{a, b, c\}\), \(\tau_1 = \{X, \emptyset, \{a\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{b\}\}\). Then the subset \{b\} is \((1, 2)\)-Rmg-closed set, but not \(\tau_2\)-closed set in the bitopological space \((X, \tau_1, \tau_2)\).

**Remark 3.9:** \(\tau_j\)-pre-closed sets and \((i, j)\)-Rmg-closed sets are independent as seen from the following example.
Example 3.10: Let \( X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\} \). Then the subset \( \{a, c\} \) is (1, 2)-RMG-closed set, but not \( \tau_2 \)-pre-closed sets in the bitopological space \( (X, \tau_1, \tau_2) \).

Example 3.11: Let \( X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, a\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, a\}\} \). Then the subset \( \{b\} \) is \( \tau_2 \)-pre-closed set, but not (1, 2)-RMG-closed sets in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.12: \( \tau_1 \)-mildly-g-closed sets and \( (i, j) \)-RMG-closed sets are independent as seen from the following example.

Example 3.13: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subsets \( \{b, d\} \) and \( \{a, b, d\} \) are \( \tau_2 \)-mildly-g-closed set, but not (1, 2)-RMG-closed sets in the bitopological space \( (X, \tau_1, \tau_2) \).

Example 3.14: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subset \( \{b\} \) is (1, 2)-RMG-closed set, but not \( \tau_2 \)-mildly-g-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.15: \( (i, j) \)-g-closed sets and \( (i, j) \)-RMG-closed sets are independent as seen from the following example.

Example 3.16: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subset \( \{c\} \) is (1, 2)-RMG-closed set, but not \( \tau_1 \)-g-closed set. Also the subsets \( \{b, d\} \) and \( \{a, b, d\} \) are (1, 2)-g-closed sets, but not (1, 2)-RMG-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.17: \( (i, j) \)-g*-closed sets and \( (i, j) \)-RMG-closed sets are independent as seen from the following example.

Example 3.18: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subset \( \{c\} \) is (1, 2)-RMG-closed set, but not (1, 2)-g*-closed set. Also the subsets \( \{b, d\} \) and \( \{a, b, d\} \) are (1, 2)-g*-closed sets, but not (1, 2)-RMG-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Theorem 3.19: If \( A \) be a \( (i, j) \)-RMG-closed subset of \( (X, \tau_1, \tau_2) \), then \( A \) is \( (i, j) \)-wg-closed, but converse is not true.

Proof: Let \( A \) be a \( (i, j) \)-RMG-closed subset of \( (X, \tau_1, \tau_2) \). Let \( G \subseteq O(X, \tau_i) \) be such that \( A \subseteq G \). Since \( O(X, \tau_i) \subseteq RGO(X, \tau_i) \), we have \( G \subseteq RGO(X, \tau_i) \). Then by hypothesis, \( \tau_1 \)-cl(\( \tau_1 \)-int(\( A \)) \) \( \subseteq G \). Therefore \( A \) is \( (i, j) \)-mildly-closed.

Example 3.20: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subsets \( \{b, d\} \) and \( \{a, b, d\} \) are (1, 2)-wg-closed set, but not (1, 2) RMG-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.21: (i, j)-RMG-closed sets and (i, j)-rw-closed sets are independent as seen from the following example.

Example 3.22: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subsets \( \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\} \) and \( \{a, b, d\} \) are (1, 2)-rw-closed sets, but not (1, 2)-RMG-closed sets. Also the subset \( \{c\} \) is (1, 2)-RMG-closed set, but not (1, 2)-rw-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.23: (i, j)-RMG-closed sets and (i, j)-rg-closed sets are independent as seen from the following example.

Example 3.24: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then the subsets \( \{a, b\}, \{a, c\}, \{b, d\}, \{a, b, c\} \) and \( \{a, b, d\} \) are (1, 2)-rg-closed sets, but not (1, 2)-RMG-closed sets. Also the subset \( \{c\} \) is (1, 2)-RMG-closed set, but not (1, 2)-rg-closed set in the bitopological space \( (X, \tau_1, \tau_2) \).

Remark 3.25: From the above discussion and know results we have the following implications.
Remark 3.26: The intersection of two (i, j)-RMG-closed sets is generally not a (i, j)-RMG-closed set as seen from the following example.

Example 3.27: Let \( X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\} \). Then the subsets \{a, c\} and \{b, c\} are (1, 2)-RMG-closed sets, but \{a, c\} \cap \{b, c\} = \{c\} is not a (1, 2) RMG-closed set in bitopological space \((X, \tau_1, \tau_2)\).

Remark 3.28: The union of two (i, j)-RMG-closed sets is generally not a (i, j)-RMG-closed set as seen from the following example.

Example 3.29: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \). Then the subsets \{b\} and \{c\} are (1, 2)-RMG-closed sets, but \{b\} \cup \{c\} = \{b, c\} is not a (1, 2)-RMG-closed set in bitopological space \((X, \tau_1, \tau_2)\).

Remarks 3.30: The family \( D_{RMG}(X, \tau_1, \tau_2) \) is generally not equal to the family \( D_{RMG}(X, \tau_2, \tau_1) \) as seen from the following example.

Example 3.31: Let \( X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \( \tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \). Then \( D_{RMG}(X, \tau_1, \tau_2) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \( D_{RMG}(X, \tau_2, \tau_1) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \). Therefore \( D_{RMG}(X, \tau_1, \tau_2) \neq D_{RMG}(X, \tau_2, \tau_1) \).

Theorem 3.32: If \( \tau_1 \subseteq \tau_2 \) and \( RGO(X, \tau_1) \subseteq RGO(X, \tau_2) \) in \((X, \tau_1, \tau_2)\), then \( D_{RMG}(X, \tau_1, \tau_2) \subseteq D_{RMG}(X, \tau_2, \tau_1) \).

Proof: Let \( A \) be a \((\tau_2, \tau_1)\)-RMG-closed set and \( G \) be an \( \tau_1 \)-rg-open set containing \( A \). By the assumption \( \tau_1 \subseteq \tau_2 \) it follows that \( G \) is \( \tau_2 \)-rg-open set containing \( A \) and \( \tau_2 \)-cl(\( \tau_1 \)-int(\( A \))) \subseteq \tau_2 \)-cl(\( \tau_1 \)-int(\( A \))). Then \( \tau_2 \)-cl(\( \tau_1 \)-int(\( A \))) \subseteq G \) and \( A \) is \((\tau_1, \tau_2)\)-RMG-closed.

Theorem 3.33: Let \( i, j \in \{1, 2\} \). For each point \( x \) of \((X, \tau_i, \tau_j)\), a singleton \( \{x\} \) is \( \tau_i \)-rg-closed or \( \{x\} \) is \((\tau_j, \tau_i)\)-RMG-closed.

Proof: Suppose \( \{x\} \) is not \( \tau_i \)-rg-closed. Then \( \{x\} \) is not \( \tau_i \)-rg-open. Therefore a \( \tau_i \)-rg-open set containing \( \{x\} \) is \( X \) only. Also \( \tau_j \)-cl(\( \tau_i \)-int(\( \{x\} \))) \subseteq X \). Hence \( \{x\} \) is \((\tau_i, \tau_j)\)-RMG-closed.

Theorem 3.34: If a subset \( A \) of \( X \) is \((i, j)\)-RMG-closed in \((X, \tau_i, \tau_j)\) if and only if \( \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \subseteq U^c \). That is \( U \cap \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \subseteq U^c \). Hence \( U \cap \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \subseteq U^c \).

Proof: Suppose that \( A \) is a \((i, j)\)-RMG-closed set in \((X, \tau_i, \tau_j)\). We prove the the result by contradiction. Let \( U \) be \( \tau_i \)-rg-closed set such that \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \) and \( U \neq \emptyset \). Then \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \) and \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \). Therefore \( U \cap \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \) is \( \tau_i \)-rg-closed set and \( A \) is \((i, j)\)-RMG-closed, \( \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \subseteq U^c \). That is \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \) and \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \). Hence \( U \subseteq \tau_j \)-cl(\( \tau_i \)-int(\( A \))) \).
Theorem 3.35: In a bitopological space $(X, \tau_1, \tau_2)$, $\text{RGO}(X, \tau_j) \subset \{ F \subset X : F^c \in \tau_j \}$ if and only if every subset of $(X, \tau_1, \tau_j)$ is a $(i, j)$-RMG-closed set.

Proof: Suppose that $\text{RGO}(X, \tau_j) \subset \{ F \subset X : F^c \in \tau_j \}$. Let $A$ be any subset of $X$. If $G \in \text{RGO}(X, \tau_j)$ then $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) - A$ does not contain any non-empty $\tau_1$-rg-closed set in $(X, \tau_1, \tau_j)$.

Conversely, assume that $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) - A$ contains no empty $\tau_1$-rg-closed set. Let $A \subseteq U$, $U$ is a $\tau_1$-rg-closed set. Suppose that $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) - U^c$ is not contained in $U$. Then $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \cap U^c$ is a non empty $\tau_1$-rg-closed set and contained in $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) - A$, which is contradiction. Therefore $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \subseteq U$. Hence $A$ is $(i, j)$-RMG-closed in $(X, \tau_1, \tau_j)$.

Theorem 3.36: If $A$ is a $(i, j)$-RMG-closed set and $A \subset B \subset \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A))$, then $B$ is $(i, j)$-RMG-closed.

Proof: Let $U$ be a $\tau_1$-rg-open set such that $B \subseteq U$. As $A$ is $(i, j)$-RMG-closed set and $A \subseteq C$, we have $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \subseteq U$. Now $B \subseteq \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A))$ which implies $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(B)) \subseteq \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(\{ \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \})) = \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(B)) \subseteq U$. Thus $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(B)) \subseteq U$. Therefore $B$ is $(i, j)$-RMG-closed set.

Theorem 3.37: Let $A \subset Y \subset X$ and suppose that $A$ is $(i, j)$-RMG-closed in $(X, \tau_1, \tau_2)$. Then $A$ is $(i, j)$-RMG-closed relative to $Y$ provided $Y$ is $\tau_1$-open set.

Proof: $\tau_{1-Y}$ be the restriction of $\tau_1$ to $Y$. Let $G \subseteq Y$ be a $\tau_{1-Y}$-open set such that $A \subseteq G$. Since $A \subseteq Y \subset X$ and $Y$ is $\tau_1$-open. By the Lemma 3.26 [12], $G$ is $\tau_{1-Y}$-open. Since $A$ is $(i, j)$-RMG-closed, $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \subseteq G$. That is $Y \cap \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \subseteq Y \cap G = G$. Also $Y \cap \tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) = \tau_{1-Y} - \text{cl}(\tau_{1-Y} \setminus \text{int}(A))$. Thus $\tau_{1-Y} - \text{cl}(\tau_{1-Y} \setminus \text{int}(A)) \subseteq G$. Hence $A$ is $(i, j)$-RMG-closed in $Y$.

Theorem 3.38: In a bitopological space $(X, \tau_1, \tau_2)$, if $\text{RGO}(X, \tau_j) = \{ X, \emptyset \}$, then every subset of $(X, \tau_1, \tau_2)$ is $(i, j)$-RMG-closed.

Proof: Let $\text{RGO}(X, \tau_j) = \{ X, \emptyset \}$ in a bitopological space $(X, \tau_1, \tau_2)$. Let $A$ be any subset of $X$. To prove that $A$ is an $(i, j)$-RMG-closed. Suppose $A \neq \emptyset$. Then $A$ is $(i, j)$-RMG-closed. Suppose $A \neq \emptyset$, then $X$ is only $\tau_1$-rg-open set and $\tau_1 - \text{cl}(\tau_1 \setminus \text{int}(A)) \subseteq X$. Hence $A$ is $(i, j)$-RMG-closed set.

4. $(i, j)$-Rmg-Open Sets And Some Of Their Properties.

In this section, we introduce $(i, j)$-RMG-open sets in bitopological spaces and study some of their properties.

Definition 4.1: Let $i, j \in \{ 1, 2 \}$ be fixed integers. In a bitopological space $(X, \tau_1, \tau_2)$, a subset $A \subset X$ is said to be $(\tau_1, \tau_j)$-Regular Mildly Generalized open (briefly, $(i, j)$-RG-open) if $A^c$ is $(i, j)$-Rmg-closed. We denote the family of all $(i, j)$-Rmg-open sets in a bitopological space $(X, \tau_1, \tau_2)$ by $D^{\tau_1, \tau_2}(i, j, \tau_1, \tau_j)$ or $D^{\tau_1, \tau_2}(i, j)$.

Theorem 4.2: In bitopological space $(X, \tau_1, \tau_2)$, we have the following.

(i) Every $(i, j)$-pre-open set is $(i, j)$-Rmg-open set but not conversely.

(ii) Every $(i, j)$-Rmg-open set is $(i, j)$-mildly g-open set but not conversely.

(iii) Every $\tau_1$-open set is $(i, j)$-Rmg-open set but not conversely.

(iv) Every $(i, j)$-Rmg-open set is $(i, j)$-wg-open set but not conversely.

Proof: The proof follows from the Theorems 3.3, 3.5, 3.7 and 3.19.

Example 4.3: Let $X = \{ a, b, c, d \}$, $\tau_1 = \{ X, \emptyset, \{ a \}, \{ b, c \}, \{ a, b, c \} \}$ and $\tau_2 = \{ X, \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, b, c \} \}$. Then the subsets $\{ c \}$ and $\{ b, c \}$ are $(1, 2)$-Rmg-open sets, but not $(1, 2)$-pre-open sets in the bitopological space $(X, \tau_1, \tau_2)$. 

[32]
Example 4.4: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then the subsets $\{c\}$ and $\{a, c\}$ are $(1, 2)$-mildly $g$-open sets, but not $(1, 2)$-RMG-open sets in the bitopological space $(X, \tau_1, \tau_2)$.

Example 4.5: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}\}$. Then the subsets $\{b\}, \{a, b\}$ and $\{a, c\}$ are $(1, 2)$-RMG-open sets, but not $\tau_2$-open sets in the bitopological space $(X, \tau_1, \tau_2)$.

Example 4.6: Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b, a, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then the subsets $\{a, b\}$, $\{a, b, c\}$ and $\{b, d\}$ are $(1, 2)$-wg-open sets, but not $(1, 2)$-RMG-open sets in the bitopological space $(X, \tau_1, \tau_2)$.

Remark 4.7: The intersection of two $(i, j)$-RMG-open sets is generally not an $(i, j)$-RMG-open sets as seen from the following example.

Example 4.8: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b, a, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, a, c\}\}$. Then the subsets $\{a, b, d\}$ and $\{a, c, d\}$ are $(1, 2)$-RMG-open sets, but $\{a, b, d\} \cap \{a, c, d\} = \{a, d\}$ is not $(1, 2)$-RMG-open set in the bitopological space $(X, \tau_1, \tau_2)$.

Remark 4.9: The union of two $(i, j)$-RMG-open sets is generally not an $(i, j)$-RMG-open sets as seen from the following example.

Example 4.10: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b, a, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then the subsets $\{a\}$ and $\{c\}$ are $(1, 2)$-RMG-open sets, but $\{a\} \cup \{c\} = \{a, c\}$ is not $(1, 2)$-RMG-open set in the bitopological space $(X, \tau_1, \tau_2)$.

Remarks 4.11: The family $D_{RMG}^*(X, \tau_1, \tau_2)$ is generally not equal to the family $D_{RMG}^*(X, \tau_2, \tau_1)$ as seen from the following example.

Example 4.12: Let $X = \{a, b, c\}, \tau_1 = \{X, \emptyset, \{b, a, c\}\}, \tau_2 = \{X, \emptyset, \{a, c\}\}$. Then $D_{RMG}^*(X, \tau_1, \tau_2) = \{X, \emptyset, \{b\}, \{a, c\}, \{b, c\}\}$ and $D_{RMG}^*(X, \tau_2, \tau_1) = \{X, \emptyset, \{a\}, \{b, a\}\}$. Therefore $D_{RMG}^*(X, \tau_1, \tau_2) \neq D_{RMG}^*(X, \tau_2, \tau_1)$.

Theorem 4.13: A subset $A$ of $(X, \tau_1, \tau_2)$ is $(i, j)$-RMG-open if and only if $F \subseteq \tau_1 \text{int}(\tau_1 \text{cl}(A))$, where $F$ is $\tau_1$-rg-closed set and $F \subseteq A$.

Proof: Suppose that $F$ is $\tau_1$-rg-closed set, $F \subseteq A$ and $A^c \subseteq G$. Then $G \subseteq A$ and $G^c$ is $\tau_1$-rg-closed. Thus $G^c \subseteq \tau_1 \text{int}(\tau_1 \text{cl}(A))$ and $[\tau_1 \text{int}(\tau_1 \text{cl}(A))]^c \subseteq G$. It follows that $\tau_1 - \text{int}(\tau_1 \text{cl}(A))^c \subseteq G$ and hence $A^c$ is $(i, j)$-RG-closed. Hence $A$ is $(i, j)$-RMG-open.

Conversely, Suppose that $A$ is $(i, j)$-RMG-open, $F \subseteq A$ and $F$ is $\tau_1$-rg-closed set. Then $F^c$ is $\tau_1$-rg-open and $A^c \subseteq F^c$. Therefore $\tau_1 - \text{cl}(\tau_1 - \text{int}(A^c)) \subseteq F^c$ and hence $[\tau_1 - \text{int}(\tau_1 - \text{cl}(A))]^c \subseteq F^c$. Thus $F \subseteq \tau_1 \text{int}(\tau_1 \text{cl}(A))$, since $\tau_1 - \text{cl}(\tau_1 - \text{int}(A^c)) = [\tau_1 - \text{int}(\tau_1 - \text{cl}(A))]^c$.

Theorem 4.14: Let $A$ and $G$ be two subsets of a bitopological space $(X, \tau_1, \tau_2)$. If the set $A$ is $(i, j)$-RMG-open, then $G = X$ whenever $G$ is $\tau_1$-rg-open and $\tau_1 \text{int}(\tau_1 \text{cl}(A)) \cup A^c \subseteq G$.

Proof: Let $A$ be $(i, j)$-RMG-open. $G$ be the $\tau_1$-rg-open and $\tau_1 \text{int}(\tau_1 \text{cl}(A)) \cup A^c \subseteq G$. Then $G^c \subseteq \tau_1 - \text{int}(\tau_1 - \text{cl}(A)) \cup A^c \subseteq G^c$. Since $A^c$ is $(i, j)$-RMG-closed and $G^c$ is $\tau_1$-rg-open, by the Theorem 3.34, it follows that $G^c = \emptyset$. Therefore $G = X$.

The converse of the above theorem need not be true as seen from the following example.

Example 4.15: Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{a, b, c\}\}$. If $A = \{a, c\}$ then only $\tau_1$-rg-open set containing $\tau_2 \text{int}(\tau_1 \text{cl}(A)) \cup A^c$ is $X$. But $A$ is not $(1, 2)$-RMG-open set in $(X, \tau_1, \tau_2)$.

Theorem 4.16: If a subset $A$ of $(X, \tau_1, \tau_2)$ is $(i, j)$-RMG-closed, then $\tau_1 \text{cl}(\tau_1 \text{int}(A)) - A$ is $(i, j)$-RMG-open.

Proof: Let $A$ be $(i, j)$-RMG-closed subset in $(X, \tau_1, \tau_2)$. Let $F$ be a $\tau_1$-rg-open set such that $F \subseteq \tau_1 \text{cl}(\tau_1 \text{int}(A)) - A$. By Theorem 3.34, $F = \emptyset$. Therefore $F \subseteq \tau_1 \text{int}(\tau_1 \text{cl}(\tau_1 \text{cl}(\tau_1 \text{int}(A)) - A))$ and by Theorem 4.13, $\tau_1 \text{int}(\tau_1 \text{cl}(A))$ is $(i, j)$-RMG-open.

The converse of the above theorem need not be true as seen from the following example.

Example 4.17: For a subset $A = \{b\}$ in $X$. $X = \{a, b, c, d\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{c, d\}\}$. Then $\tau_1 \text{cl}(\tau_1 \text{int}(A)) - A = \{a, b\} - \{b\}$ is $(1, 2)$-RMG-open but $A = \{b\}$ is not $(1, 2)$-RMG-closed.
**Theorem 4.18:** If $\tau_i$-int$(\tau_j$-$cl(A)) \subseteq B \subseteq A$ and A is $(i, j)$-RMG-open in $(X, \tau_j, \tau_2)$, Then B is $(i, j)$-RMG-open.

**Proof:** Let $F$ be $\tau_i$-$rg$-open such that $F \subseteq B$. Now $F \subseteq B \subseteq A$. That is $F \subseteq A$. Since $F$ is $(i, j)$-RMG-open, by Theorem 4.13, $F \subseteq \tau_i$-$int(\tau_j$-$cl(A))$. By hypothesis $\tau_i$-$int(\tau_j$-$cl(A)) \subseteq B$. Therefore $\tau_i$-$int(\tau_j$-$cl(A)) \subseteq \tau_i$-$int(\tau_j$-$cl(B))$. That is $\tau_i$-$int(\tau_j$-$cl(A)) \subseteq \tau_i$-$int(\tau_j$-$cl(B))$ and hence $F \subseteq \tau_i$-$int(\tau_j$-$cl(B))$. Again by Theorem 4.13, B is $(i, j)$-RMG-open set in $(X, \tau_j, \tau_2)$.

**Corollary 4.19:** Let A and B be subsets of a space $(X, \tau_j, \tau_2)$. If B is $(i, j)$-RMG-open and $A \supseteq \tau_i$-$int(\tau_j$-$cl(B))$, Then $A \cap B$ is $(i, j)$-RMG-open.

**Proof:** Let B be $(i, j)$-RMG-open and $A \supseteq \tau_i$-$int(\tau_j$-$cl(B))$. That is $\tau_i$-$int(\tau_j$-$cl(B)) \subseteq A$. Then $\tau_i$-$int(\tau_j$-$cl(B)) \subseteq A \cap B$. Also $\tau_i$-$int(\tau_j$-$cl(B)) \subseteq A \cap B$ is $(i, j)$-RMG-open. By Theorem 4.18, $A \cap B$ is $(i, j)$-RMG-open.

**Theorem 4.20:** Every singleton point set in a space $(X, \tau_j, \tau_2)$ is either $(i, j)$-RMG-open or $\tau_i$-$rg$-closed.

**Proof:** Let $(X, \tau_j, \tau_2)$ be a bitopological space. Let $x \in X$. To prove that $\{x\}$ is either $(i, j)$-RMG-open or $\tau_i$-$rg$-closed. That is to prove $X-\{x\}$ is either $(i, j)$-RMG-closed or $\tau_i$-$rg$-closed. Which follows from the Theorem 3.33.

5.(\tau_i, \tau_j$)-RMG-Neighbourhoods and some of their properties.

**Definition 5.1:** In a bitopological space $(X, \tau_j, \tau_2)$. A subset N of X is said to be $(\tau_i, \tau_j$)-RMG-neighbourhood (briefly, $(\tau_i, \tau_j$)-RMG-nhd) of a point $x \in X$ if there exists $(\tau_i, \tau_j$)-RG-open set G such that $x \in G \subseteq N$.

**Example 5.2:** $X=\{a, b, c, d\}$, $\tau_1=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\} \}$ and $\tau_2=\{X, \emptyset, \{a\}, \{b, c\}, \{a, c\} \}$; then $(\tau_i, \tau_j$)-RMG-closed sets are $X, \emptyset, \{c\}, \{a, c\}, \{a, b, c\}$ and $(\tau_i, \tau_j$)-RMG-open sets are $X, \emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, d\}, \{a, b, c\}$. Let $c \in \{b, c\}$ is $(\tau_i, \tau_j$)-RG-open.

\[ \vdash \{a, b, c\} \text{ is a } (\tau_i, \tau_j$)-RMG-neighbourhood of c.\]

The collection of all $(\tau_i, \tau_j$)-RMG-neighbourhoods of $x \in X$ is denoted by $(\tau_i, \tau_j$)-RMG-$N(x)$.

**Theorem 5.3:** If N be a subset of a bitopological space $(X, \tau_j, \tau_2)$ is $(\tau_i, \tau_j$)-RMG-open set, then N is $(\tau_i, \tau_j$)-RG-nhd of each of its points.

**Proof:** In a bitopological space $(X, \tau_j, \tau_2)$. Let N be $(\tau_i, \tau_j$)-RMG-open set.

\[ \forall x \in N \text{ there exists } (\tau_i, \tau_j$)-RG-open set N such that $x \in N \subseteq N. \]

\[ \Rightarrow N \text{ is } (\tau_i, \tau_j$)-RMG-neighbourhood of x.

**Remark 5.4:** (i, j)-RMG-nhd of point $x \in X$ need not be a $(i, j)$-nhd of x in X as seen from the following example.

**Example 5.5:** Let $X=\{a, b, c, d\}$, $\tau_1=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\} \}$ and $\tau_2=\{X, \emptyset, \{a\}, \{b, c\}, \{a, c\} \}$. $\text{D}^\text{RMG}(\tau_i, \tau_j)=\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c\} \}$. Let $c \in \{b, c\}$ is $(i, j)$-RG-nhd of c since there exists a $(i, j)$-RG-open set $\emptyset \subseteq \{b, c\}$. Also the set the set $\{b, c\}$ is $(i, j)$-RG-nhd of c, since there exists a $(i, j)$-RG-open set $\{c\}$ such that $c \in \{c\}$. However $\{b, c\}$ is not $(i, j)$-RG-open set in X.

**Theorem 5.6:** Let $(X, \tau_i, \tau_j)$ be a bitopological space.

(i) $\forall x \in X, (\tau_i, \tau_j)-\text{RG} - N(x) \neq \emptyset$.

(ii) If $N \in (\tau_i, \tau_j) - \text{RG} - N(x)$ and $N \subseteq M \Rightarrow M \in (\tau_i, \tau_j) - \text{RG} - N(x)$.

(iii) If $N \in (\tau_i, \tau_j) - \text{RG} - N(x) \Rightarrow \exists M \in (\tau_i, \tau_j) - \text{RG} - N(x)$ such that $M \subseteq N$ and $M \in (\tau_i, \tau_j) - \text{RG} - N(y) \forall y \in M$.

**Proof:**

(i) Since X is an $(\tau_i, \tau_j$)-RMG-open set, it is a $(\tau_i, \tau_j$)-RG-nhd of $\forall x \in X$. Hence there exists at least one $(\tau_i, \tau_j$)-RG-neighbourhood G for every $x \in X$. Therefore $(\tau_i, \tau_j)$-RG-$N(x)$ \(\neq \emptyset\).

(ii) If $\text{RG}(\tau_i, \tau_j)$-RG-$N(x)$ and $x \subseteq M$ then there exists an $(\tau_i, \tau_j)$-RG-open set G such that $x \in G \subseteq N$, since $N \subseteq M$, $x \in G \subseteq M$ and M is $(\tau_i, \tau_j)$-RMG-nhd of x. Hence M is an $(\tau_i, \tau_j)$-RG-neighbourhood of x. Therefore $M \in (\tau_i, \tau_j)$-RG-$N(x)$.

(iii) If $\text{RG}(\tau_i, \tau_j)$-RG-$N(x)$, then there exists an $(\tau_i, \tau_j)$-RG-open set M such that $x \in M \subseteq N$. Since M is an $(\tau_i, \tau_j)$-RMG-open set, then it is $(\tau_i, \tau_j)$-RG-neighbourhood of each of its points. Therefore $M \in (\tau_i, \tau_j)$-RG-$N(y)$ $\forall y \in M$.

**Theorem 5.7:** Every $\tau_j$-neighbourhood of a point x of a bitopological space $(X, \tau_1, \tau_2)$ is a $(\tau_i, \tau_j)$-RMG-neighbourhood of the point x.
Proof: \((X, \tau_1, \tau_2)\) is a bitopological space and \(x \in X\). \(N\) is a \(\tau_j\)-neighbourhood of \(x\).
\(\Rightarrow\) \exists a \(\tau_j\)-open set \(G\) such that \(x \in G \subseteq N\).
\(\Rightarrow\) \exists \((\tau_i, \tau_j)\)-RMG-open set \(G\) such that \(x \in G \subseteq N\).
\(\Rightarrow\) \(N\) is a \((\tau_i, \tau_j)\)-RMG-neighbourhood of the point \(x\).

The converse of the above theorem need not be true as seen from the following example.

**Example 5.8:** Let \(X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \) and \(\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\). \((\tau_i, \tau_j)\)-RMG-closed sets are \(X, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\). \((\tau_i, \tau_j)\)-RMG-open sets are \(X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\) and \((\tau_i, \tau_j)\)-RMG-open sets are \(X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\) where \(\{a, b\}\) is \((\tau_i, \tau_j)\)-RMG-open. Therefore \(\{a, b, c\}\) is \((\tau_i, \tau_j)\)-RMG-neighbourhood of \(c\). \(N = \{b, c\}\) is a \((\tau_i, \tau_j)\)-RMG-neighbourhood of \(c\). But there does not exist an open set \(G\) in \(\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\) such that \(x \in G \subseteq N\). Therefore \(N\) is not \(\tau_2\)-neighbourhood of \(c\).

**Theorem 5.9:** If \(F\) is a \((\tau_i, \tau_j)\)-RMG-closed subset of a bitopological space \((X, \tau_1, \tau_2)\), then \(\forall x \in F^c\) there exists an \((\tau_i, \tau_j)\)-RMG-open set \(N\) containing \(x\) such that \(N \cap F = \emptyset\).

Proof: \(F\) is a \((\tau_i, \tau_j)\)-RMG-closed subset of a bitopological space \((X, \tau_1, \tau_2)\)
\(\Rightarrow\) \(F^c\) is a \((\tau_i, \tau_j)\)-RMG-open subset of \(X\).
\(\Rightarrow\) \(F^c\) is a \((\tau_i, \tau_j)\)-neighbourhood of each of its points.
\(\Rightarrow\) There exists a \((\tau_i, \tau_j)\)-RMG-open set \(N\) such that \(x \in N \subseteq F^c\) \(\forall x \in F\).
\(\Rightarrow\) There exists a \((\tau_i, \tau_j)\)-RMG-open set \(N\) containing \(x\) such that \(N \cap F = \emptyset\).

**6. \((\tau_i, \tau_j)\)-Rmg-Closure and Some of Their Properties.**

**Definition 6.1:** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(i, j \in \{1, 2\}\) be fixed integers. For each subset \(E\) of \(X\), define \((\tau_i, \tau_j)\)-RMG-\(cl(E) = \bigcap \{A : \emptyset \subseteq A \subseteq \text{D}_{\text{RMG}}(i, j)\}\) and is denoted by \((i, j)\)-RMG-\(cl(E)\).

**Theorem 6.2:** Let \(A\) and \(B\) be subset of \((X, \tau_1, \tau_2)\). Then
(i) \((i, j)\)-RMG-\(cl(X) = X\) and \((i, j)\)-RMG-\(cl(\emptyset) = \emptyset\).
(ii) \(A \subseteq (i, j)\)-RMG-\(cl(A)\).
(iii) If \(B\) is any \((i, j)\)-RMG-closed set containing \(A\), \(then (i, j)\)-RMG-\(cl(A) \subseteq B\).

Proof: Follows from the Definition 6.1.

**Theorem 6.3:** Let \(A\) and \(B\) be subset of \((X, \tau_1, \tau_2)\) and \(i, j \in \{1, 2\}\) be the fixed integers. If \(A \subseteq B\), then \((i, j)\)-RMG-\(cl(A) \subseteq (i, j)\)-RMG-\(cl(B)\).

Proof: Let \(A \subseteq B\). By Definition 6.1, \((i, j)\)-RMG-\(cl(B) = \bigcap \{F : B \subseteq F \subseteq \text{D}_{\text{RMG}}(i, j)\}\). If \(B \subseteq F \subseteq \text{D}_{\text{RMG}}(i, j)\), since \(A \subseteq B\), \(A \subseteq B \subseteq \text{D}_{\text{RMG}}(i, j)\), we have \((i, j)\)-RMG-\(cl(A) \subseteq F\). Therefore \((i, j)\)-RMG-\(cl(A) \subseteq \bigcap \{F : B \subseteq F \subseteq \text{D}_{\text{RMG}}(i, j)\} = (i, j)\)-RMG-\(cl(B)\). That is \((i, j)\)-RMG-\(cl(A) \subseteq (i, j)\)-RMG-\(cl(B)\).

**Theorem 6.4:** Let \(A\) be a subset of \((X, \tau_1, \tau_2)\). If \(\tau_1 \subseteq \tau_2\) and \(\text{RGO}(X, \tau_1) \subseteq \text{RGO}(X, \tau_2)\), then \((1, 2)\)-RMG-\(cl(A) \subseteq (2, 1)\)-RMG-\(cl(A)\).

Proof: By definition 6.1, \((1, 2)\)-RMG-\(cl(A) = \bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(1, 2)\}\). Since \(\tau_1 \subseteq \tau_2\), by Theorem 3.32, \(\text{D}_{\text{RMG}}(2, 1) \subseteq \text{D}_{\text{RMG}}(1, 2)\). Therefore \(\bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(2, 1)\} \subseteq \bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(1, 2)\}\). That is \((1, 2)\)-RMG-\(cl(A) \subseteq \bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(2, 1)\} = (2, 1)\)-RMG-\(cl(A)\) Hence \((1, 2)\)-RMG-\(cl(A) \subseteq (2, 1)\)-RMG-\(cl(A)\).

**Theorem 6.5:** Let \(A\) be a subset of \((X, \tau_1, \tau_2)\) and \(i, j \in \{1, 2\}\) be fixed integers, then \(A \subseteq (i, j)\)-RMG-\(cl(A) \subseteq \tau_j\)-\(cl(A)\).

Proof: By definition 6.1, it follows that \(A \subseteq (i, j)\)-RMG-\(cl(A)\). Now to prove that \((i, j)\)-RMG-\(cl(A) \subseteq \tau_j\)-\(cl(A)\). By definition of closure, \(\tau_j\)-\(cl(A) = \bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(\tau_j)\}\). Since \(\tau_1 \subseteq \tau_2\), by Theorem 3.32, \(\text{D}_{\text{RMG}}(\tau_1) \subseteq \text{D}_{\text{RMG}}(\tau_2)\). Therefore \(\bigcap \{F : A \subseteq F \subseteq \text{D}_{\text{RMG}}(\tau_2)\} = (\tau_j\)-\(cl(A)\). Hence \((i, j)\)-RMG-\(cl(A) \subseteq \tau_j\)-\(cl(A)\).

**Remark 6.6:** Containment relation in above theorem may be proper as seen from the following example.

**Example 6.7:** Let \(X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}\) and \(\tau_2 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}\). Then \(\tau_2\)-closed sets are \(X, \emptyset, \{a\}, \{a, d\}, \{a, c\}, \{a, b, c\}\) and \((1, 2)\)-RMG-closed sets are \(X, \emptyset, \{c\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\). Take \(A = \{a, c\}\). Then
Theorem 6.8: Let A be a subset of \((X, \tau_i, \tau_j)\) and \(i, j \in \{1, 2\}\) be fixed integers. If \(A\) is \((i, j)\)-RMG-closed, then \((i, j)\)-RMG-cl-closed \(= A\).

**Proof:** Let \(A\) be a \((i, j)\)-RMG-closed subset of \((X, \tau_i, \tau_j)\). We know that \(A \subseteq (i, j)\)-RMG-cl \(A\). Also \(A \subseteq A\) and \(A\) is \((i, j)\)-RMG-closed. By the Theorem 6.2(iii), \((i, j)\)-RMG-cl \(A\) \(\subseteq A\). Hence \((i, j)\)-RMG-cl \(A\) \(= A\).

Remark 6.9: The converse of the above Theorem 6.7 need not be true as seen from the following example.

**Example 6.10:** Let \(X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \emptyset, \{a, b\}, \{c, d\}\}\). Then \((1, 2)\)-RMG-closed sets are \(X, \emptyset, \{a\}, \{b\}, \{c, d\}\). Take \(A = \{a\}\). Now \((1, 2)\)-RMG-cl \(A\) \(= X\cap \{a\}\) \(\cap \{b\}\) \(\cap \{a, c, d\} = \{a\}\), but \(A\) is not a \((1, 2)\)-RMG-closed.

**Theorem 6.11:** And B are subsets of a bitopological space \((X, \tau_i, \tau_j)\), then \((\tau_i, \tau_j)\)-RMG-cl \((A \cup B) \subseteq (\tau_i, \tau_j)\)-RMG-cl \((A\cup B)\).

**Proof:** Let \(A\) and \(B\) are subset of \((X, \tau_i, \tau_j)\). Clearly \(A \cup B \subseteq A\) and \(B \subseteq A\).

\[ \Rightarrow (\tau_i, \tau_j)\)-RMG-cl \(A \cup B \subseteq (\tau_i, \tau_j)\)-RMG-cl \(A\), \(B\) \(\subseteq (\tau_i, \tau_j)\)-RMG-cl \(A\) \(\cup B\) \(\subseteq (\tau_i, \tau_j)\)-RMG-cl \(A\cup B\).

**Remark 6.12:** \((\tau_i, \tau_j)\)-RMG-cl \(A\) \(\subseteq (\tau_i, \tau_j)\)-RMG-cl \(A\) \(\cup (\tau_i, \tau_j)\)-RMG-cl \(A\).

**Example 6.13:** Let \(X = \{a, b, c, d\}, \tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{X, \emptyset, \{a, b\}, \{c, d\}\}\). Then \((1, 2)\)-RMG-closed sets are \(X, \emptyset, \{a\}, \{b\}, \{c, d\}\). Let \(A = \{a\}\) and \(B = \{b\}\). Then \(A \cup B = \{a, b\}\), \((\tau_i, \tau_j)\)-RMG-cl \(A\) \(= \{a\}\), and \((\tau_i, \tau_j)\)-RMG-cl \(B\) \(= \{b\}\).

**Theorem 14.1:** If \(A\) is a subset of a bitopological space \((X, \tau_i, \tau_j)\), then \((\tau_i, \tau_j)\)-RMG-cl \((A \cap B) \subseteq (\tau_i, \tau_j)\)-RMG-cl \((A\cap B)\).

**Proof:** Since \((\tau_i, \tau_j)\)-RMG-cl \(A\) \(= (\tau_i, \tau_j)\)-RMG-cl \((A)\) \(\cap (\tau_i, \tau_j)\)-RMG-cl \((B)\).

**Theorem 15.1:** If \(A\) and \(B\) are subsets of a bitopological space \((X, \tau_i, \tau_j)\), then \((\tau_i, \tau_j)\)-RMG-cl \((A \cap B) \subseteq (\tau_i, \tau_j)\)-RMG-cl \((A\cap B)\).

**Proof:** Let \(A\) and \(B\) are subset of \((X, \tau_i, \tau_j)\). Clearly \(A \cap B \subseteq A\) and \(B \subseteq B\).

\[ \Rightarrow (\tau_i, \tau_j)\)-RMG-cl \((A \cap B) \subseteq (\tau_i, \tau_j)\)-RMG-cl \((A\cap B)\).

**Theorem 6.16:** A is a nonempty subset of a bitopological space \((X, \tau_i, \tau_j)\), \(x \in X\) and \(x \notin (\tau_i, \tau_j)\)-RMG-cl \((A)\) if and only if \(\forall \tau_i, \tau_j\)-RMG-open set \(V\) containing \(x\).

**Proof:** Let \(A\) be a nonempty subset of a bitopological space \((X, \tau_i, \tau_j)\). Let \(x \in X\) and \(x \notin (\tau_i, \tau_j)\)-RMG-cl \((A)\). To prove the result by contradiction. Suppose \(\exists \) a \((\tau_i, \tau_j)\)-RMG-open set \(V\) containing \(x\) such that \(V \cap A = \emptyset\). Then \(\forall \tau_i, \tau_j\)-RMG-closed set and so \((\tau_i, \tau_j)\)-RMG-cl \((A) \subseteq \tau_i, \tau_j\)-RMG-cl \((X)\). This shows that \(x \notin (\tau_i, \tau_j)\)-RMG-cl \((A)\), which is a contradiction. Hence \(\forall \tau_i, \tau_j\)-RMG-open set \(V\) containing \(x\).

Conversely, let \(\forall \tau_i, \tau_j\)-RMG-open set containing \(x\). To prove \(x \notin (\tau_i, \tau_j)\)-RMG-cl \((A)\). We prove the result by contradiction. Suppose that \(x \notin (\tau_i, \tau_j)\)-RMG-cl \((A)\).

Then there exists a \((\tau_i, \tau_j)\)-RMG-closed subset \(F\) containing \(A\) such that \(x \in F\). Then \(x \in \tau_i, \tau_j\)-RMG-open. Also \((X, \tau_i, \tau_j)\)-RMG-open.

**Theorem 6.17:** If \(A\) is a subset of a bitopological space \((X, \tau_i, \tau_j)\). Then \((\tau_i, \tau_j)\)-RMG-cl \(A\) \(\subseteq (\tau_i, \tau_j)\)-pre-cl \(A\).

**Proof:** Let \(A\) be a subset of \((X, \tau_i, \tau_j)\). By the definition of \((\tau_i, \tau_j)\)-pre-closure, \((\tau_i, \tau_j)\)-pre-closure \(A\) \(= \cap \{F \subseteq X: \tau_i, \tau_j\}-RMG\)-closed set \(X\). \(A \subseteq \tau_i, \tau_j\)-pre-closed set \(X\). Therefore \(A \subseteq \tau_i, \tau_j\)-pre-closed set \(X\).
Theorem 6.18: If $A$ be subset of a bitopological space $(X, \tau_i, \tau_j)$.

(i) $(\tau_i, \tau_j)$-mildly-$g$-$cl(A) \subseteq (\tau_i, \tau_j)$-$RMG$-$cl(A)$

(ii) $(\tau_i, \tau_j)$-$wg$-$cl(A) \subseteq (\tau_i, \tau_j)$-$RMG$-$cl(A)$

Proof: (i) Let $A$ be a subset of $(X, \tau_i, \tau_j)$. By the definition of $(\tau_i, \tau_j)$-$RMG$-closure, $(\tau_i, \tau_j)$-$RMG$-$cl(A) = \bigcap \{F: A \subseteq F \in D_{RMG}(i, j)\}$. If $A \subseteq F \in D_{RMG}(i, j)$, then $A \subseteq F \in (\tau_i, \tau_j)$-mildly-$g$-$closed$. Because as every $(\tau_i, \tau_j)$-$RMG$-closed set is $(\tau_i, \tau_j)$-mildly-$g$-$closed$, that is $(\tau_i, \tau_j)$-mildly-$g$-$cl(A) \subseteq F$. Therefore $(\tau_i, \tau_j)$-mildly-$g$-$cl(A) \subseteq \bigcap \{F: A \subseteq F, (\tau_i, \tau_j)$-$RMG$-$cl(A)\}$. Therefore $(\tau_i, \tau_j)$-mildly-$g$-$cl(A) \subseteq (\tau_i, \tau_j)$-$RMG$-$cl(A)$.

(ii) Similarly (ii) results may be proved.

7. $(\tau_i, \tau_j)$-$RMG$-Interior and Some of Their Properties.

Definition 7.1: Let $A$ be a subset of a bitopological space $(X, \tau_i, \tau_j)$. We define the $(\tau_i, \tau_j)$-$RMG$-interior of $A$ to be the union of all $(\tau_i, \tau_j)$-$RMG$-open sets contained in $A$ and is denoted by $(\tau_i, \tau_j)$-$RMG$-$int(A)$.

Theorem 7.2: Let $A$ and $B$ be subsets of bitopological space $(X, \tau_i, \tau_j)$, then

(i) $(i,j)$-$RMG$-$int(X)=X$ and $(i,j)$-$RMG$-$int(\emptyset)=\emptyset$.

(ii) $(i,j)$-$RMG$-$int(A) \subseteq A$.

(iii) If $B$ is any $(i,j)$-$RMG$-open set contained in $A$. Then $B \subseteq (i,j)$-$RMG$-$int(A)$.

(iv) If $A$ is $(i,j)$-$RMG$-open, Then $(i,j)$-$RMG$-$int(A)=A$.

Proof: (i) Since $X$ and $\emptyset$ are $(i,j)$-$RMG$-open sets. By the definition $7.1$, $(i,j)$-$RMG$-$int(X)=X$ and $(i,j)$-$RMG$-$int(\emptyset)=\emptyset$.

(ii) If $F=\{G: G \subseteq A \subseteq \tau_i \text{ and } G \subseteq A \}$, then $(i,j)$-$RMG$-$int(A) = \bigcup G_{G \subseteq F}$ and $G \subseteq A$, $\forall G \in F \cup G_{G \subseteq F} \subseteq A$. That is $(i,j)$-$RMG$-$int(A) \subseteq A$.

(iii) Let $B$ be any $(i,j)$-$RMG$-open set such that $B \subseteq A$. Let $x \in B$. Then since $B$ is an $(i, j)$-$RMG$-open set contained in $A$, $x$ is an $(i,j)$-$RMG$-interior point of $A$. That is $x \in (i,j)$-$RMG$-$int(A)$. Hence $B \subseteq (i,j)$-$RMG$-$int(A)$.

(iv) If a subset $A$ of space $X$ is $(i,j)$-$RMG$-open subset of $X$. We know that $(i,j)$-$RMG$-$int(A) \subseteq A$. Also, $A$ is $(i,j)$-$RMG$-open set contained in $A$. From above result (ii), $A \subseteq (i, j)$-$RMG$-$int(A)$. Hence $(i,j)$-$RMG$-$int(A)=A$.

Theorem 7.3: Let $A$ and $B$ be subsets of bitopological space $(X, \tau_i, \tau_j)$.

(i) If $A \cup B$, then $(i,j)$-$RMG$-$int(A) \subseteq (i,j)$-$RMG$-$int(B)$.

(ii) $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A) \cap (i,j)$-$RMG$-$int(B)$.

(iii) $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A \cup B)$.

Proof: (i) Let $A$ and $B$ be subsets of bitopological space $(X, \tau_i, \tau_j)$. Then $A \subseteq B$.

$(i,j)$-$RMG$-$open\{F = (i,j)$-$RMG$-$open \cap A \subseteq F \subseteq B \text{ and } F \in (i,j)$-$RMG$-$open\}$

$(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A \cup B)$.

(ii) We know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. We have by above result (i), $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A)$ and $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(B)$. This implies that $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A) \cap (i,j)$-$RMG$-$int(B)$.

(iii) Let $A$ and $B$ be subsets of bitopological space $(X, \tau_i, \tau_j)$. We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. We have $(i,j)$-$RMG$-$int(A \cap B) \subseteq (i,j)$-$RMG$-$int(A \cup B)$ and $(i,j)$-$RMG$-$int(B) \subseteq (i,j)$-$RMG$-$int(A \cup B)$. This implies that $(i,j)$-$RMG$-$int(A \cup B) \subseteq (i,j)$-$RMG$-$int(A \cap B)$.

Theorem 7.4: If $A$ is a subset of a bitopological space $(X, \tau_i, \tau_j)$, then

(i) $(i,j)$-$RMG$-$int(A) \subseteq (i,j)$-$mildly-g$-$int(A)$.

(ii) $(i,j)$-$RMG$-$int(A) \subseteq (i,j)$-$wg$-$int(A)$.

Proof: (i) $A$ is a subset of a bitopological space $(X, \tau_i, \tau_j)$. Let $x \in (i,j)$-$RMG$-$int(A) \Rightarrow x \in \bigcup \{G \subseteq X: G \text{ is } (i,j)$-$RMG$-$open, G \subseteq A\}$

$(i,j)$-$mildly-g$-$open \text{ such that } x \in G$ and $G \subseteq A$.

$(i,j)$-$wg$-$open \text{ such that } x \in G$ and $G \subseteq A$.

$(i,j)$-$mildly-g$-$int(A)$.

$x \in (i,j)$-$mildly-g$-$int(A)$.  

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Hence (i, j)-RMG-int(A) ⊆ (i, j)-mildly-g-int(A).

(ii) Similarly these results may be proved.

Theorem 7.5: If A is a subset of a bitopological space $(X, \tau_1, \tau_2)$, then (i, j)-pre-int(A) ⊆ (i, j)-RMG-int(A).

Proof: A is a subset of a bitopological space $(X, \tau_1, \tau_2)$.
Let $x \in (i, j)$-pre-int(A) ⇒ $x \in U \{G \subseteq X: G$ is (i, j)-pre-open, $G \subseteq A\}$
⇒ there exists a $(i, j)$-pre-open set $G$ such that $x \in G$ and $G \subseteq A$, as every $(i, j)$-pre-open set is (i, j)-RMG-open set in $X$.
⇒ there exists a $(i, j)$-RMG-open set $G$ such that $x \in G$ and $G \subseteq A$.
⇒ $x \in U\{G: G$ is (i, j)-RMG-open, $G \subseteq A\}$.
⇒ $x \in (i, j)$-RMG-int(A).
Hence $(i, j)$-pre-int(A) ⊆ $(i, j)$-RMG-int(A).

Theorem 7.6: Let $A$ be any subset of $(X, \tau_1, \tau_2)$. then
(i) $X-(i, j)$-RMG-int(A) = $(i, j)$-RMG-cl(X-A).
(ii) $(i, j)$-RMG-int(A) = $X-(i, j)$-RMG-cl(X-A).
(iii) $(i, j)$-RMG-int(X-A) = $(i, j)$-RMG-cl(A).
(iv) $X-(i, j)$-RMG-cl(A) = $(i, j)$-RMG-int-(X-A).

Proof: (i) Let $x \in X-(i, j)$-RMG-int(A). Then $x \not\in (i, j)$-RMG-int(A). That is every (i, j)-RMG-open set $U$ containing $x$ is such that $U \subseteq A$. That is every (i, j)-RMG-open set $U$ containing $x$ such that $U \cap (X-A) \neq \emptyset$. By Theorem 6.16, $x \notin (i, j)$-RMG-cl(X-A) and Therefore $X-(i, j)$-RMG-int(A) ⊆ $(i, j)$-RMG-cl(X-A).
Conversely, $x \not\in (i, j)$-RMG-cl(X-A). Then by Theorem 6.16, every $(i, j)$-RMG-open set $U$ containing $x$ is such that $U \cap (X-A) \neq \emptyset$. That is every $(i, j)$-RMG-open set $U$ containing $x$ such that $U \subseteq A$. This implies by Definition of $(i, j)$-RMG-interior of $A$, $x \notin (i, j)$-RMG-int(A). That is $x \in X-(i, j)$-RMG-int(A) and $(i, j)$-RMG-cl(X-A) ⊆ X-(i, j)-RMG-int(A). Thus $X-(i, j)$-RMG-int(A) = $(i, j)$-RMG-cl(X-A).

(ii) Follows by taking compliments in (i).
(iv) Follows by replacing $A$ by $X-A$ in (i).
(v) Follows by result (i).

8. References