On certain subclasses of analytic multivalent functions involving generalized integral operator

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Abstract
In this paper several new subclasses of analytic functions which are defined by means of a generalized integral operator have been introduced. Next, inclusion properties for these subclasses are established. Many interesting applications are also discussed.

Keywords: analytic functions, univalent and multivalent functions, starlike functions, convex functions, differential subordination, hadamard product or convolution

Introduction
Let $A_p$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N})$$

which are analytic in the open unit disk $U = \{z: z \in \mathbb{C} \text{ and } |z| < 1 \}$. Also let $S_p(\alpha)$ and $K_p(\alpha)$ denote, respectively, the subclasses of $A_p$ consisting of $p$-valent functions which are starlike and convex of order $\alpha$ in $U$ with $0 \leq \alpha < p$. In particular $S_p(0) = S_p$ and $K_p(0) = K_p$ are the well-known subclasses of $p$-valent starlike and $p$-valent convex functions in $U$, respectively.

Given two functions $f$ and $g$, which are analytic in $U$ with $f(0) = g(0)$, the function $f$ is said to be subordinate to $g$ in $U$ if there exists a function $w$, analytic in $U$, such that $w(0) = 0$, $|w(z)| < 1$ (for $z \in U$), and $f(z) = g(w(z))$ (for $z \in U$).

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We denote this subordination by $f(z) \prec g(z)$ in $U$.

We also observe that $f(z) \prec g(z)$ in $U$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ whenever $g$ is univalent in $U$.

Let $M$ be the class of analytic functions $\phi(z)$ in $U$ normalized by $\phi(0) = 1$, and let $H$ be the subclass of $M$ consisting of those functions $\phi$ which are univalent in $U$ and for which $\phi(U)$ is convex and $R \{\phi(z)\} > 0$ (for $z \in U$).

We define the following subclasses $S_p^*(\phi)$ and $K_p(\phi)$ for $\phi \in H$ by

$$S_p^*(\phi) = \left\{ f: f \in A_p \text{ and } \frac{z f'(z)}{f(z)} \prec p \phi(z) \text{ in } U \right\}$$

$$K_p(\phi) = \left\{ f: f \in A_p \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec p \phi(z) \text{ in } U \right\}.$$
Obviously

1. \( s_p \left( \frac{1+z}{1-z} \right) = S^*_p \),

2. \( k_p \left( \frac{1+z}{1-z} \right) = K_p \),

3. \( s^*_p \left( \frac{1+Az}{1+Bz} \right) = S^*_p [A,B] \) \((-1 < B < A \leq 1)\),

4. \( k_p \left( \frac{1+Az}{1+Bz} \right) = K_p [A,B] \) \((-1 < B < A \leq 1)\).

We also have \( S^*_p [1,-1] = S^*_p \) and \( K^*_p [1,-1] = K_p \). For \( p = 1 \), the above reduced classes \( S^*_p [A,B] \) and \( K [A,B] \) were investigated by Janowski [7] and Goel and Mehrok [4].

Clearly

\[
f (z) \in k_p (\phi) \iff zf^\prime (z) \in S^*_p (\phi).
\]

Further suppose that

\[
h_p [(\alpha_q); (\beta_i); z] = z^p F_q \left( \alpha_1, ..., \alpha_q; \beta_1, ..., \beta_i; z \right)
\]

\[
= z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q, \beta_i)} (n) z^n
\]

\[(q \leq r + 1; \alpha_i \in R; \beta_j \in R \setminus \mathbb{Z}_0^+; \mathbb{Z}_0^+ = \{0, -1, -2, ...\}; i = 1, ..., q; j = 1, ..., r; z \in U) \]

where \( \phi P \) is the generalized hypergeometric function and

\[
B_p^{(\alpha_q, \beta_i)} (n) = \frac{(\alpha_1)_{n-p} (\alpha_2)_{n-p} ... (\alpha_q)_{n-p}}{(\beta_1)_{n-p} (\beta_2)_{n-p} ... (\beta_i)_{n-p} (n-p)_i}.
\]

Corresponding to the function \( h_p [(\alpha_q); (\beta_i); z] \), Dziok and Srivastava [2, p.3, Eq.(3)] introduced a linear operator \( H_{p,\alpha_q,\beta_i} \) defined by the convolution

\[
H_{p,\alpha_q,\beta_i} f (z) = h_p [(\alpha_q); (\beta_i); z] * f (z)
\]

or equivalently by

\[
H_{p,\alpha_q,\beta_i} f (z) = z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q, \beta_i)} (n) a_n z^n \quad (z \in U).
\]

Here * stands for the convolution of two analytic multivalent functions \( f \) and \( g \) of the form

\[
f (z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g (z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \quad (a_n, b_n \geq 0, \quad p \in \mathbb{N})
\]

and is defined by

\[
(f \ast g) (z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.
\]

The linear operator \( H_{p,\alpha_q,\beta_i} \) includes various other linear operators considered earlier by Hohlov [6], Carlson-Shaffer [1], Goyal and Bhagtani [5], Ruscheweyh [9] etc.

Next by using the operator \( H_{p,\alpha_q,\beta_i} \) we introduce the following classes of analytic functions for \( \phi \in H; f \in A_p; \alpha_q > -1 \) and \( \beta_i \geq 1 \)

\[
s_{p,\alpha_q,\beta_i} (\phi) = \{ f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_i} f (z) \in S^*_p (\phi) \}
\]

“107”
\[ k_{p,\alpha,\beta, r} (\phi) = \{ f : f \in A_p \text{ and } H_{p,\alpha,\beta, r} f (z) \in k_p (\phi) \} \]

We also note that
\[ f (z) \in k_{p,\alpha,\beta, r} (\phi) \iff z f '(z) \in S_{p,\alpha,\beta, r} (\phi) \quad \ldots (1.7) \]

In particular, we set
\[ S_{p,s,i,2} \left( \frac{1+z}{1-z} \right) = S_{p,s,i}, \]
\[ S_{p,\lambda,v,p,\mu} \left( \frac{1+Az}{1+Bz} \right) = S_{p,\lambda,v,p,\mu} [A,B] \quad (-1 \leq B < A \leq 1), \]
\[ k_{p,\lambda,v,i,\mu} \left( \frac{1+Az}{1+Bz} \right) = K_{p,\lambda,v,i,\mu} [A,B] \quad (-1 \leq B < A \leq 1), \]

and for \( p = 1 \), we have
\[ \text{if } r = s = n \text{ then } S_{n,2} \left( \frac{1+z}{1-z} \right) = S_n. \]
\[ \text{if } \nu = \lambda \text{, then } S_{\lambda,\nu} \left( \frac{1+Az}{1+Bz} \right) = S_{\lambda,\nu} [A,B] \quad (-1 \leq B < A \leq 1) \]

And
\[ K_{\lambda,\nu} \left( \frac{1+Az}{1+Bz} \right) = K_{\lambda,\nu} [A,B] \quad (-1 \leq B < A \leq 1). \]

2. Inclusion properties involving \( H_{p,\alpha,\beta, r} \)

The following result will be required in our investigation:

**Lemma (Eenigenburg et al. [13]).** Let \( h \) be convex univalent in \( U \) with \( h(0) = 1 \) and \( R \{ \beta h(z) + \gamma \} > 0 \quad (\beta, \gamma \in C) \)

If \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \), then
\[ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < h(z) \text{ in } U \]

implies that \( p(z) < h(z) \) in \( U \).

Our first inclusion theorem is stated as:

**Theorem 1.** Let \( \alpha_q > -1 \) and \( \beta_r \geq 1 \). Then
\[ S_{p,\alpha,\beta, r} (\phi) \subset S_{p,\alpha,\beta, r} (\phi) \subset S_{p,\alpha+1,\beta, r} (\phi) \quad (\phi \in H) \]

**Proof.** First of all we show that
\[ S_{p,\alpha,\beta, r} (\phi) \subset S_{p,\alpha,\beta, r} (\phi) \quad (\phi \in H; \quad \alpha_q > -1 \text{ and } \beta_r \geq 1) \]

Let \( f(z) \) \( S_{p,\alpha,\beta, r} (\phi) \) and set
\[ \frac{z \left( H_{p,\alpha,\beta, r} f(z) \right)}{H_{p,\alpha,\beta, r} f(z)} = p \theta(z), \quad \ldots (2.1) \]

where \( \theta(z) = 1 + c_1 z + c_2 z^2 + \ldots \).

Obviously \( \theta(z) \) is analytic in \( U \) and \( \theta(z) \neq 0 \) for all \( z \in U \).

Applying (1.21) in (2.1), we obtain
\[ (\beta_r + p - 1) \frac{H_{p,\alpha,\beta, r+1} f(z)}{H_{p,\alpha,\beta, r} f(z)} = p \theta(z) + \beta_r - 1 \quad \ldots (2.2) \]

By using the logarithmic differentiation on both sides of (2.2) and multiplying with \( z \), we have
In Theorem 1, we obtain

\[
\frac{z \{H_{p,a_1,\beta_1,1} f(z)\}}{H_{p,a_1,\beta_1} f(z)} = p \vartheta(z) + \frac{p z \vartheta'(z)}{p \vartheta(z) + \beta_r - 1}
\] 

...(2.3)

Since \( \beta_r \geq 1 \), \( \phi(z) \in H \) and \( f(z) \in s_{p,a_1,\beta_1}^* (\phi) \) from (2.3), we see that

\[ R \{ p \phi(z) + \beta_r - 1 \} > 0 \quad (z \in U) \]

and

\[ p \vartheta(z) + \frac{p z \vartheta'(z)}{p \vartheta(z) + \beta_r - 1} \leq p \phi(z) \text{ in } U \]

Thus by using the Lemma and (2.1) we observe that

\[ p \Theta(z) \leq p \phi(z) \text{ in } U \]

so that

\[ f(z) \in s_{p,a_1,\beta_1}^* (\phi) \]

This implies that

\[ s_{p,a_1,\beta_1+1}^* (\phi) \subseteq s_{p,a_1,\beta_1}^* (\phi) \]

To prove the second part, let \( f(z) \in s_{p,a_1,\beta_1}^* (\phi) \) \( (\alpha_q>1 \text{ and } \beta_r \geq 1) \) and put

\[
\frac{z \{H_{p,a_1,\beta_1+1,1} f(z)\}}{H_{p,a_1,\beta_1+1} f(z)} = p \Psi(z)
\]

where \( \Psi(z) = 1 + d_1 z + d_2 z^2 + \ldots \) is analytic in \( U \) and \( \Psi(z) \neq 0 \) for all \( z \in U \). Now by using arguments similar to those detailed above, it follows that

\[ p \Psi(z) > p \phi(z) \text{ in } U, \]

which implies that \( f(z) \in s_{p,a_1,\beta_1+1}^* (\phi) \). Hence we conclude that

\[ s_{p,a_1,\beta_1+1}^* (\phi) \subseteq s_{p,a_1,\beta_1}^* (\phi) \subseteq s_{p,a_1,\beta_1,1}^* (\phi) \quad (\phi \in H) \]

which completes the proof of Theorem 1

Remark. By putting

\[ q = 2, r = 2; \alpha_1 = r, \alpha_2 = s, \beta_1 = 1, \beta_2 = 2, (r, s, t \in N_0), \mu = 2 \text{ and } \phi(z) = \frac{1 + z}{1 - z}, z \in U \text{ in Theorem 1, we obtain} \]

\[ s_{p,2s}^* \subseteq s_{p,2s+1,1}^* \]

Further if \( p = 1 \) and \( r = s = n \) then \( S_{n}^* \subseteq S_{n+1}^* \) which was asserted earlier by Noor [8].

Theorem 2. Let \( \alpha_q > 1 \) and \( \beta_r \geq 1 \),

\[ k_{p,a_1,\beta_1+1} (\phi) \subseteq k_{p,a_1,\beta_1} (\phi) \subseteq k_{p,a_1+1,\beta_1} (\phi) \quad (\phi \in H) \]

Proof. Applying (1.7) and Theorem 1, we observe that

\[ f(z) \in k_{p,a_1,\beta_1+1} (\phi) \Leftrightarrow H_{p,a_1,\beta_1+1} f(z) \in k_{p} (\phi) \]

\[ \Rightarrow z (H_{p,a_1,\beta_1+1} f(z)) \in s_{p}^* (\phi) \Leftrightarrow H_{p,a_1,\beta_1+1} (zf'(z)) \in s_{p}^* (\phi) \]

\[ \Rightarrow zf'(z) \in s_{p,a_1,\beta_1+1}^* (\phi) \Leftrightarrow zf'(z) \in s_{p,a_1,\beta_1}^* (\phi) \Leftrightarrow H_{p,a_1,\beta_1} (zf'(z)) \in s_{p}^* (\phi) \]

\[ \Rightarrow z (H_{p,a_1,\beta_1} f(z)) \in s_{p}^* (\phi) \Leftrightarrow H_{p,a_1,\beta_1} f(z) \in k_{p} (\phi) \Leftrightarrow f(z) \in k_{p,a_1,\beta_1} (\phi) \]

and

\[ f(z) \in k_{p,a_1,\beta_1} (\phi) \Leftrightarrow zf'(z) \in s_{p,a_1,\beta_1}^* (\phi) \]

"109"
\[ z f'(z) \in S^*_{p,\alpha_q+1,\beta}(\phi) \iff z (H_{p,\alpha_q+1,\beta} f(z))' \in S^*_p(\phi) \]
\[ \iff H_{p,\alpha_q+1,\beta} f(z) \in k_p(\phi) \iff f(z) \in k_{p,\alpha_q+1,\beta}(\phi) \]

which evidently proves Theorem 2.

Taking
\[ \phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1; \ z \in U) \]

in Theorems 1 and 2, we have

**Corollary.** Let \( \alpha_q > -1 \) and \( \beta > 1 \) and \(-1 \leq B < A \leq 1\). Then
\[ S^*_{p,\alpha_q,\beta_1}[A,B] \subset S^*_{p,\alpha_q,\beta}[A,B] \subset S^*_{p,\alpha_q+1,\beta}[A,B] \]
and
\[ K_{p,\alpha_q,\beta_1}[A,B] \subset K_{p,\alpha_q,\beta}[A,B] \subset K_{p,\alpha_q+1,\beta}[A,B] \]

**References**


"110"