Stability of the pexiderized cauchy functional equation in \((n, \beta)\)-Hilbert space

PM Parimala

Abstract
The concept of pexiderized Cauchy functional equation we obtain the general solution for the new additive functional equation. We prove some results of pexiderized Cauchy functional equation. Let \( f \) be the mapping from a linear space \( x \) into a complete random normed space \( y \). The Cauchy functional equation and the Cauchy-pexiderized functional equation, Hilbert space are generalized and their solution are determined.

Keywords: Pexiderized cauchy functional equation, \((n, \beta)\)-Hilbert space

Introduction

Let us being by restarting and solving Cauchy’s functional equation.

Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function satisfying

\[
f(x + y) = f(x) + f(y) \quad \text{for all real} \]

We show that there exists a real number that \( f(x) = ax \) for all \( x \in \mathbb{R} \) it is straightforward to show by mathematical induction (1.2.1) implies

\[
f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n) \quad \text{for all} \ x_1, x_2, \ldots, x_n \in \mathbb{R}
\]

A special case of this is found by setting \( x_1 = x_2 = \cdots = x_n \) say then (1.3.2) becomes

\[
f(nx) = f(mt) \quad \text{for all positive integer} \ n \ \text{and for all real} \ x
\]

Let \( x = \left(\frac{m}{n}\right) t \) where \( m \) and \( n \) are positive integers then \( nx = mt \) so,

\[
f(nx) = f(mt)
\]

\[
 nf(x) = mf(t)
\]

\[
 nf\left(\frac{m}{n} t\right) = mf(t)
\]

But this can be written as

\[
f\left(\frac{m}{n} t\right) = \frac{m}{n} f(t) \quad \text{for all} \ t \in \mathbb{R}
\]

Then we have proved that

\[
f(qt) = qf(t) \quad \text{for all real value of} \ t \ \text{and all rational of} \ q
\]
We can extended (1.2.5) to include $q = 0$ in the following way returning to (1.2.1) we see that

\[
f(y) = f(y + 0) = f(y) + f(0)
\]

So $f(0) = 0$ from this we immediately have $f(0t) = 0f(t)$ therefore (1.2.5) is true for all non-negative rational $q$.

Once again, returning to (1.2.1) we obtain

\[
0 = f(0)
\]

\[
= f(q + (-q))
\]

\[
= f(q) + f(-q)
\]

Therefore $f(-q) = -f(q)$ from this we get for $q < 0$

\[
f(qt) = f(-q)t
\]

\[
= -f((-q)t)
\]

\[
= -(-q)f(t)
\]

\[
= qf(t)
\]

Therefore $f(qt) = qf(t)$ for all real value of $t$ and all rational of $q$  \hspace{1cm} (1.2.6)

Now suppose we substitute $t = 1$ into (1.2.6) letting $f(1) = a$ we deduce that $f(a) = qa$ for all rational number $q$

We summarization what has been discovered so far in the following proposition.

**Pexider’s Equations**

**The Additive Pexider Functional Equation**

\[
f(x + y) = h(x) + g(y) \quad (x,y) \in k
\]

(1.4.1)

It is well known (1.4.1) that the continuous solutions of this functional equation on the domain $k$ are of the form

\[
f(t) = c \cdot t + a + b
\]

(1.4.2)

\[
h(t) = c \cdot t + b
\]

(1.4.3)

\[
g(t) = c \cdot t + a
\]

(1.4.4)

Where $a = g(0)$, $b = h(0)$

We want to get the solutions for this equation on the curve $\Gamma_1 \cup \Gamma_2$. By placing $(x, y)$ that are in the domain $\Gamma_1 \cup \Gamma_2$ in (1.4.1) we get seven equations

\[
f(t) = g(\beta(t)) + h(\alpha(t))
\]

(1.4.5)

\[
f(1) = g(-\beta(t)) + h(\alpha(t))
\]

(1.4.6)

\[
f(t) = g(\alpha(t)) + h(\beta(t))
\]

(1.4.7)

\[
f(-1) = g(\beta(t)) + h(-\alpha(t))
\]

(1.4.8)

\[
f(t) = g(t) + h(0)
\]

(1.4.9)

\[
f(t) = g(0) + h(t)
\]

(1.4.10)

\[
f(0) = g(0) + h(0)
\]

(1.4.11)

Where $\alpha(t) = \frac{t+1}{2}$, $\beta(t) = \frac{t-1}{2}$

By placing (1.4.9), (1.4.10) and (1.4.11) in the first four equations we obtain four new equations

\[
f(t) = f(\beta(t)) + f(\alpha(t)) - f(0) \quad (by\ (1.4.5))
\]

\[
f(1) = f(-\beta(t)) + f(\alpha(t)) - f(0) \quad (by\ (1.4.6))
\]

\[
f(t) = f(\alpha(t)) + f(\beta(t)) - f(0) \quad (by\ (1.4.7))
\]

\[
f(-1) = f(\beta(t)) + f(-\alpha(t)) - f(0) \quad (by\ (1.4.8))
\]

Now we define a new function

\[
m(t) = f(t) - f(0) \Rightarrow f(t) = f(t) + f(0)
\]

By placing $f(t) = f(t) + f(o)$ in (3) we get

\[
m(t) = m(\alpha(t)) + m(\beta(t))
\]

(1.4.12)

\[
m(1) = m(-\beta(t)) + m(\alpha(t))
\]

(1.4.13)

\[
m(-1) = m(\beta(t)) + m(-\alpha(t))
\]

(1.4.14)
This is Cauchy’s additive functional equation, and its continuous solutions are of the form \( m(t) = c \cdot t \)
That gives us these solutions of the equation (4.1.1) on the domain \( \Gamma_1 \cup \Gamma_3 \)

\[
\begin{align*}
  f(t) &= c \cdot t + a + b \\
  h(t) &= c \cdot t + b \\
  g(t) &= c \cdot t + a \\
  \text{Where } b = h(0) \quad a = g(0)
\end{align*}
\]

So we can conclude that there are no new continuous solutions on the curve. \( \Gamma_1 \cup \Gamma_3 \)
We seek the solutions for this equation on the curve \( \Gamma_2 \). To this end, place \( y = x \) in the equation (4.1.1) to obtain

\[
f(2x) = h(x) + g(x), x \in [\frac{-1}{2}, \frac{1}{2}]
\]

For this equation we didn’t find the general solutions but there are new solutions, for example

* \( f(t) = 1, h(t) = \cos^2 t, g(t) = \sin^2 t \)
* \( f(t) = \cos(2t), h(t) = \cos^2 t, g(t) = -\sin^2 t \)
* \( f(t) = \sin(2t), h(t) = g(t) = \cos(t) \cdot \sin(t) \)

Note that these solutions are \( c^e \), so no smoothness condition will preserve the set of solutions (1.4.2). However in Cauchy’s additive equation we do not obtain new solutions when we add the condition that the function will be \( c^e \) or differentiable at 0.

In this section, we investigate the stability of the pexiderized Cauchy functional equation in \( (n, \beta) \)-Hilbert spaces.

**Theorem**

Let \( X \) be a vector space and \( Y \) be a complete \( (n, \beta) \)-Hilbert space with \( 0 < \beta \leq 1 \). Let \( \varphi : X_2 \rightarrow [0, \infty) \) be a function satisfying

\[
\Phi(x) = \sum_{i=1}^{\infty} 2 - i\beta \left( \Phi(2i - 1, 0) + \Phi(0, 2i - 1) + \Phi(2i - 1, 2i - 1) \right) < \infty \quad (4.1.1)
\]

And

\[
\lim_{m \to \infty} 2 - m\beta \Phi(2mx, 2my) = 0 \quad (4.1.2)
\]

For all \( x, y \in X \), \( \psi : Y \times \cdots \times Y \rightarrow (n-1) \rightarrow [0, \infty) \) is a function. If mappings \( f, g, h : X \rightarrow Y \) satisfy the inequality

\[
\|f(x+y) - g(x) - h(y), z_1, \ldots, z_{n-1}\| \leq \varphi(x, y) \psi(z_1, \ldots, z_{n-1}) \quad (4.1.3)
\]

For all \( x, y \in X \) and \( z_1, \ldots, z_{n-1} \in Y \), then there exists a unique additive mapping \( A : X \rightarrow Y \) satisfying

\[
\|f(x) - A(x), z_1, \ldots, z_{n-1}\| \leq \Phi(x), z_1, \ldots, z_{n-1} \quad (4.1.4)
\]

\[
\|g(x) - A(x), z_1, \ldots, z_{n-1}\| \leq \Phi(x), z_1, \ldots, z_{n-1} + 2\|h(0), z_1, \ldots, z_{n-1}\| \quad (4.1.5)
\]

\[
\|h(x) - A(x), z_1, \ldots, z_{n-1}\| \leq \Phi(x), z_1, \ldots, z_{n-1} + 2\|g(0), z_1, \ldots, z_{n-1}\| \quad (4.1.6)
\]

for all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \).

**Proof**

Putting \( y = x \) in inequality (4.1.3), we get

\[
\|f(2x) - g(x) - h(x), z_1, \ldots, z_{n-1}\| \leq \varphi(x, x) \psi(z_1, \ldots, z_{n-1}) \quad (4.2.7)
\]

for all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \). Putting \( y = 0 \) in inequality (4.1.3), we get

\[
\|f(x) - g(x) - h(0), z_1, \ldots, z_{n-1}\| \leq \varphi(x, 0) \psi(z_1, \ldots, z_{n-1}) \quad (4.2.8)
\]

For all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \). It then follows from (4.1.8) that

\[
\|f(x) - g(x), z_1, \ldots, z_{n-1}\| \leq \varphi(x, 0) \psi(z_1, \ldots, z_{n-1}) + \varphi(0, x) \psi(z_1, \ldots, z_{n-1}) \quad (4.1.9)
\]

for all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \). Putting \( x = 0 \) in inequality (4.1.3), we get \( \|f(y) - g(0) - h(y), z_1, \ldots, z_{n-1}\| \leq \varphi(0, y) \psi(z_1, \ldots, z_{n-1}) \)

for all \( y \in X \) and \( z_1, \ldots, z_{n-1} \in Y \). Thus, we obtain

\[
\|f(x) - h(x), z_1, \ldots, z_{n-1}\| \leq \varphi(0, x) \psi(z_1, \ldots, z_{n-1}) + \varphi(0, x) \psi(z_1, \ldots, z_{n-1}) \quad (4.1.10)
\]

for all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \).

Let us define

\[
u(x, z_1, \ldots, z_{n-1}) = \varphi(0, x) \psi(z_1, \ldots, z_{n-1}) + \varphi(0, x) \psi(z_1, \ldots, z_{n-1}) + \varphi(0, x) \psi(z_1, \ldots, z_{n-1})
\]

Using (4.1.7), (4.1.9) and (4.1.10), we have
For all $x \in X$ and $z_1, \ldots, z_n \in Y$. Replacing $A(x)$ by $\beta \leq \lim_{m \to \infty} 2^{-m \beta} \phi(2mx, 2my) \psi(z_1, \ldots, z_n)$ we get

$$\lim_{m \to \infty} 2^{-m \beta} \phi(2mx, 2my) \psi(z_1, \ldots, z_n) = 0 \quad \text{as} \quad m \to \infty.$$
\[ A(x) = A(x) - f(x) = h(0) + g(0) + \Phi(x)\psi(z_1, \ldots, z_{n-1}) \]

By using Riesz representation theorem “If \( H \) is a Hilbert space and if \( f_y \in H^* \) then there exist a unit vector \( y \) in \( H \) such that \( f_y(x) = \langle x, y \rangle \) for every \( x \) in \( H \)

\[ \langle x, y \rangle = \langle x, z_1, \ldots, z_{n-1} \rangle \rightarrow \langle x, 0 \rangle + \langle 0, y \rangle + \langle z_1, \ldots, z_{n-1}, y \rangle \]

For all \( x \in X \) and \( z_1, \ldots, z_{n-1} \in Y \).

It remains to prove the uniqueness of \( A \).

Assume that \( A' : X \rightarrow Y \) is another additive mapping which satisfies (4.1.4). Then we have

\[ A(x) - A(x), z_1, \ldots, z_{n-1} \| \beta \| < 2 - m \| A(2mx), z_1, \ldots, z_{n-1} \| \beta \| < 2 - m \| A(x), z_1, \ldots, z_{n-1} \| \beta \| + \| h(0), z_1, \ldots, z_{n-1} \| \beta \| + \| g(0), z_1, \ldots, z_{n-1} \| \beta \| + \Phi(x)\psi(z_1, \ldots, z_{n-1}) \]

which together with Lemma implies that \( A(x) = A'(x) \) for all \( x \in X \). Using (4.1.4) and (4.1.9), we can get (4.1.5), and also using (4.1.4) and (4.1.10), we can get (4.1.6).

**Conclusion**

In this paper, we proved some results of pexiderized Cauchy functional equation. Finally we investigate the stability of pexiderized Cauchy functional equation in \((n, \beta)\) is normed space the converted to \((n, \beta)\) is Hilbert space.

**References**