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Some properties of commuting automorphism

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Abstract
Let G be a group. Definition of commuting automorphism is given. We give some properties of commuting automorphism

Keywords: Commuting automorphism, A(G)-group

1. Introduction

An automorphism g of G is called a commuting automorphism if $[g(x), x] = 1$ for all x ε G, i.e. each element x in G commutes with its image under g [5]. A(G) denotes the set of all commuting automorphisms of G. A group is said to be an A(G) group if the set of all commuting automorphisms of G forms a subgroup of Aut(G). If G is a simple non-abelian group then A(G) = 1 [5, 2], this problem was given by Herstein to the American Mathematical Monthly. For this problem, Laffey proved that A(G) =1, if G has no non-trivial abelian normal subgroups [5, 4]. Pettet found that A(G) = 1, If Z(G)=1 and G' = G. Deaconescu, Silberberg and walls gave problems about A(G). The problems were whether A(G) is always a subgroup of Aut(G), to find conditions on G st $A(G)=Aut_z(G)$. Deaconescu *et al*. gave an example of a group of order 2^5 s.t. A(G) does not form a subgroup of Aut(G). Vosooghpour, Malayeri proved that for G to be non-A(G) p-group the minimum order of G is p^5. Recently P.K. Rai gave sufficient conditions on a finite p-group G s.t. G is an A(G) - group.

2. Basic facts

2.1 [6] An automorphism φ of G is called a central automorphism If it commutes with all inner automorphisms or equivalently, if $g^{-1}\varphi(g)$ ε Z (G) ∀ g ε G.

A cyclic group is generated by a single element. We denote a cyclic group of order n by C_n. The rank of a group is denoted by d(G), which is the smallest generating set of G. The least common multiple of orders of the elements of a finite group G is called the exponent of G.

The commutator of a, b ε G is $[a, b] = a^{-1}b^{-1}ab$ and the commutator subgroup G' of G is the subgroup of G generated by all commutators of G. A maximal subgroup of G is a proper subgroup S s.t. there is no subgroup H of G s.t. $S \subset H \subset G$. The Frattini subgroup φ(G) of G is the intersection of all maximal subgroups of G. If G has no maximal subgroup then φ(G) = G. An element a ε G is called non-generator of G, If G = <a, Y> then G = <Y>. φ(G) is exactly the set of all non-generators of G. Let G be a finite group then G is nilpotent if and only of G/φ(G) is nilpotent.

[3] A finite collection of normal subgroups H_i of a group G is a normal series for G if $1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots\ldots \subseteq H_r = G$. This normal series is a central series if $H_i/H_{i-1} \subseteq Z(G/H_{i-1})$ for $1 \leq i \leq r$. A group G is nilpotent if it has a central series. Subgroups and factor groups of nilpotent groups are nilpotent.

Given any group G, we define a central series as follows. Let $Z_0 = 1$ and $Z_1 = Z(G)$. The second center Z_2 is defined to be the unique subgroup such that $Z_2/Z_1 = Z(G/Z_1)$. We continue like this, inductively defining Z_n for n > 0 so that $Z_n/Z_{n-1} = Z(G/Z_{n-1})$.

The chain of normal subgroups.

$$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots$$

Constructed in this way is called upper central series of G. The upper central series may not actually be a central series for G because it may happen that $Z_i < G$ for all i. If $Z_r = G$ for some integer r, then $\{Z_i \mid 0 \leq i \leq r\}$ is a true central series and G is nilpotent.

If G is an arbitrary nilpotent group, then G is a term of its upper central series. As $G = Z_r$ for some integer $r \geq 0$, and the smallest integer r for which this happens is called the nilpotence class of G. Non trivial abelian groups have nilpotence class 1, and for non-abelian groups of nilpotency class 2 have quotient group G/Z(G) abelian.

2.2 Lemma: ([3]) Let G be finite. Then the following are equivalent.

1. G is nilpotent
2. Every nontrivial homomorphic image of G has a nontrivial center.
3. G appears as a member of its upper central series.

2.3 Theorem. ([3]) Let G be a (not necessarily finite) nilpotent group with central series.

$1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots\ldots\ldots \subseteq H_r = G$, and let $1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \ldots\ldots\ldots$ be the upper central series of G. Then $H_i \subseteq Z_i$ for $0 \leq i \leq r$, and in particular, $Z_r = G$.

Theorem 2.4 ([3]) Let H is a subgroup of G, where G is a nilpotent group. Then $N_G(H) > H$.

Theorem 2.5 For any group G, Inn(G) is a normal subgroup of Aut(G). Further Inn(G) ≅ G/Z(G), where Z(G) denotes the centre of G.

3. Some Results on commuting automorphisms.

Theorem 3.1 ([5, 1]) Let G be a group s.t. Z(G') contains no involutions. Then A(G) is a subgroup of Aut(G) Iff commutators of elements in A(G) are central automorphisms

Theorem 3.2 ([5, 1]) If G is a group and g ε A(G) then $[G^2, g] \leq Z_2(G)$

Theorem 3.3 ([5, 4]) If g ε A(G) and x, y ε G then [g(x), y] = [x, g(y)]

Theorem 3.4 ([5, 7]) Let G be a group of nilpotency class 2. If d(G/Z(G)) = 2 then G is an A(G) group.

Theorem 3.5 ([5, 7]) For a given prime p, the minimal number of generators of a non-A(G) p-group of order p^5 and of nilpotency class 2 is equal to 1.

4. Some properties of commuting automorphisms

4.1 [f(x), x] = {e}, even If x, f(x) ∉ Z(G)

Since f(x). x = x f(x) for each x ε G

$\Rightarrow (f(x))^{-1} x^{-1} f(x). x = \{e\}$

$\Rightarrow [f(x), x] = \{e\}$

On the other hand [a, b] = {e} if a or b ε Z(G)

4.2 $(f(x))^{-1}$ commutes with x

Since f(x). x = x. f(x), $x = (f(x))^{-1} x f(x)$

$\Rightarrow x(f(x))^{-1} = (f(x))^{-1}.x$

4.3 If order of G is even, f(x) commutes with x^{-1}

because $f(x) x^{-1} = x^{-1} f(x)$, If o(x) = 2

i.e. f(x) commutes with x^{-1}

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