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The order completion method for nonlinear partial differential equations

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Abstract
In this paper we focus on the fundamentals of the Order Completion Method (OCM) for differential algebras of generalized functions. This OCM is used for various systems of nonlinear partial differential equations. Generally, outcome for generalized results of initial value problems achieved with the help of OCM. The results we find fulfill the initial condition in appropriate manner, and all the outcomes can be demonstrated in a canonical way with the use of generalized partial derivatives. The solution is based on characterization of order convergence of various sequences of quasi-continuous functions in a manner of piecewise convergence of this type of sequences. Moreover the differential algebras are corresponding to the dense algebras illustrated by Rosinger & Verneave.

Keywords: fundamentals, nonlinear partial differential equations

1. Introduction

It is a general agreement between researchers having specialization in differential equations of nonlinear type as well asindependent theory for the existence. It is also described the basic regularity of generalized results of these equations which is not even possible. Now when we consider the linear topological spaces of generalized functions to analysis the partial differential equations. So there exists two possible outcomes for the non-success of the above spaces of generalized functions to have results of huge family of linear as well as nonlinear differential equations. The first outcome tells that the spaces does not have sufficiently singular objects rather the Sobolev spaces have vice-versa results. Because for some special cases these spaces have regular, smooth and generalized solutions of differential equations. The second outcome shows, singularities which may happens in the results of nonlinear differential equations.

In the literature, when we focus on analytic nonlinear differential equations given by Cauchy-Kovalevskaia fundamentals which ensures that the existence of an analytic solution. This solution is nothing but a neighborhood of a specified non-characteristic analytic hypersurface. Thus, the result which described over the system of nonlinear equations with singularities. The generalized case of analytic function withan essential singularity at a particular point. Picard fundamentals state that the function will consistsall critical value except for every neighborhood of the singularity. The other reason of the failure of spaces of generalized functions to attain results of high families of systems of nonlinear differential equations. Although, the generalized functions with these spaces are typically expressed in terms of certain growth situations.

The elements are locally integral of the Sobolev spaces, and the Colombeau algebras of generalized functions. Where these types of functions are needed to fulfil certain polynomial kind growth conditions near singularities. The conclusive remarks while concerning the existence of results of analytic systems of nonlinear differential equations, the shortage of such type of conditions is transparent. Although, an analytic function which has an essential singularity at a particular point to grow as much as higher than the polynomial around that singularity. Furthermore, the earlier described insufficiency of the customary functional analytic methods, the order completion method (OCM), introduced in the 1994 monograph, delivers generalized outcomes of a huge family of systems of continuous nonlinear differential equations. These results are established as the elements which is nothing but the Dedekind order completion of preferable spaces of piecewise smoothfunctions.

Moreover, the results attained in this manner have been demonstrated to fulfil a fundamental blanket regularity in the sense that the results might be incorporated with Hausdorff continuous interval valued functions. The OCM was reconstructed by defining suitable uniform convergence spaces. This may have a prominent enhancement in the regularity of the generalized results achieved and into the structure of the results. This proposed work describes how these techniques defined in literature may be described in order to include initial and final(boundary) values that may be regarded as a specified system of nonlinear differential equations. This present state of art should be compared with the general linear functional analytic methods for simplifying linear and nonlinear equations, where the existence of initial & final values often leads to significant complexities, which definitely require variousmethods.

However, we have some other advantage of solving nonlinear partial differential equations by the concepts illustrated in literature. The initial as well as boundary value problems are answered with the help of the techniques that apply to the free problem. The proposed work is organized in a following manner. The next section will define some fundamental concepts related to the spaces of normal lower semi-continuous functions upon which the generalized function's spaces are constructed in third section. Later, in the section fourth the existence of generalized results of a huge family of initial value problems is demonstrated, where we also talk about the structure as well as regularity of the solutions.

2. Fundamentals of Semi-continuous Function

In section two we reveal some fundamental concepts regarding spaces of normal lower semi-continuous functions upon which the spaces of generalized functions are constructed. Specifically, these generalized function's spaces are constructed as the accomplishments of uniform convergence spaces, the elements of which are normal lower semi-continuous functions. In a manner the exposition as self-contained as possible, we thus illustrated a brief account of the spaces.

Let us assume that Θ is represented as open subset of U^g and expressed by □(Θ) the group of extended real valued function over open subset. Such that: $□(\Theta) = \left\{ v : \theta \to \bar{U} \right\}$, where $\bar{U} = U \cup \{\pm\infty\}$ is nothing but the extended real line. The lower as well as upper Baire operators I:□(Θ) $\to$ □(Θ)and R:□(Θ) $\to$ □(Θ)are defined through the followings:

$$Q(v) : \theta \ni a \mapsto \sup\{\inf\{v(b) : b \in V\} : V \in \Re\} \in \bar{U} \tag{1}$$

$$R(v) : \theta \ni a \mapsto \inf\{\sup\{v(b) : b \in V\} : V \in \Re\} \in \bar{U} \tag{2}$$

Where, $\Re$ represents the neighborhood filter at 'a' belongs to Θ. The relation of eq. 1 and 2 defines the

$$\forall v \in □(\Theta): Q(v) \le v \le R(v) \tag{3}$$

Moreover, the above mentioned operators Q, R and their compositions are idempotent as well as monotone w.r.t. the point wise order on □(Θ). So that

$\forall v \in$ □(Θ):

$$\begin{aligned} &(1)\ Q(Q(v)) = Q(v), \\ &(2)\ R(R(v)) = R(v), \\ &(3)\ (Q \circ R)((Q \circ R)(v)) = (Q \circ R)(v) \end{aligned} \tag{4}$$

and

$\forall v \in$ □(Θ):

$$v \le w \Rightarrow \begin{pmatrix} (1) Q(v) \le Q(w) \\ (2) R(v) \le R(w) \\ (3) (Q \circ R)(v) \le (Q \circ R)(w) \end{pmatrix} \tag{5}$$

A function represented as *v*belongs to □(Θ)is nothing but a normal lower semi-continuous at a∈Θ whenever

$$(Q \circ R)(v)(a) = v(a) \tag{6}$$

Here*v* is described as a normal lower semi-continuous over Θ provided that semi-continuous at all the points'a' belongs to Θ. Thus, a normal lower semi-continuous function is said to be finite approximately whenever

$\{a \in \theta : v(a) \in U\}$ is open as well as dense in Θ (7)

The group of approximate finite normal lower semi-continuous functions over Θ is described by *ML*(Θ). So, for areal continuous function over Θ is approximately finite & also normal lower semi-continuous, thus we have the following

$$\Im^0(\theta) \subseteq ML(\theta) \quad (8)$$

In the other context, every function *v*belongs to *ML*(Θ) is nothing but a continuous over a residual group. Such that

$\forall v \in ML(\Theta)$: ∃C⊂Θ of first Baire category : (9)

a∈ Θ \ C =>*v* is continuous at a.
Furthermore, the following properties of continuous functions are usefulwhich extends to *ML*(Θ):

$\forall v, w \in ML(\Theta)$: $\forall E \subseteq \theta$ dense:
$$(\forall a \in E : v(a) \leq w(a)) \Rightarrow v \leq w. \quad (10)$$

It can also be represented as w.r.t. pointwise order

$$v \leq w \Leftrightarrow \left(\forall a \in \theta : v(a) \leq w(a)\right) \quad (11)$$

The group *ML*(Θ) is nothing but a Dedekind complete lattice. In a particular case, the infimum as well as supremum of a group □⊂*ML*(Θ) is represented by the following equations:

Inf □ = $(Q \circ R)(\zeta)$ (12)

Sup □ = $(Q \circ R)(\xi)$ (13)

Where, $\Theta : \theta \ni a \mapsto \inf\{v(a) : v \in$ □ } and
$\xi : \theta \ni a \mapsto \sup\{v(a) : v \in$ □ }.

Moreover, the lattice *ML*(Θ) is known as fully distributive and defined as:

$\forall w \in ML(\Theta)$: ∀ □ ⊂*ML*(Θ):
$v_0 = \sup$ □ $\Rightarrow \sup\{\inf\{v, w\} : v \in$ □$\} = \inf\{v_0, w\}$ (14)

The characterization of order bounded sets which is useful in terms of pointwise bounded groups is called as shown below. If a group □⊂*ML*(Θ)fulfils

∃C⊂Θ of first Bairecategory: $\forall a \in \theta \setminus C :$ (15)
$\sup\{v(a) : v \in$ □$\} < \infty$

Then,

$\exists v_0 \in ML(\Theta)$: $\forall v \in$ □: $v \leq v_0$ (16)

The twofold statement for groups bounded from specified below also contains. For n∈ N ∪ {0},we focus the group
$\aleph L^n(\theta) = \{ v \in NL(\theta) \mid \exists \kappa \subset \theta$ closed nowhere dense:

$$v \in \varpi^n(\theta \setminus \kappa)\} \quad (17)$$

All the spaces $\aleph L^{n}(\theta)$ is said to be a sub-lattice of $NL(\theta)$. In a special case, $NL^{0}(\theta)$ is η-order dense in $NL(\theta)$. So, for all values of*v*belongs to $NL(\theta)$ we consist

$$\exists(\beta_n),(\omega_n)\subset\aleph L^{0}(\theta):$$
$$(1)\ \beta_n\le\beta_{n+1}\le v\le\omega_{n+1}\le\omega_n,\ n\in N$$
$$(2)\ \sup\{\beta_n:n\in N\}=v=\inf\{\omega_n:n\in N\} \quad (18)$$

The spaces of generalized functions described in literature are represented as the completions of suitable uniform convergence spaces. For this type of situation, a uniform convergence structure is illustrated over $NL^{0}(\theta)$ in a manner described below.

Definition 1: Let us consider that the Σ contains of all non-empty order intervals in $NL^{0}(\theta)$. Also, we can assume λ_0 represents the class of filters over $NL^{0}(\theta)$ × $NL^{0}(\theta)$ that fulfil the conditions given below. There exists each p belongs to N in the following manner:

$$\forall k=1,2,3,\ldots\ldots\ldots p:$$
$$\exists\sum_{j}=(Q_n^k)\subseteq\Sigma:$$
$$\exists v_k\in NL(\theta):$$
$$(1)Q_{n+1}^k\subseteq Q_n^k,n\in N,$$
$$(2)\sup\{\inf Q_n^k:n\in N\}=v_k=\inf\{\sup Q_n^k:n\in N\},$$
$$(3)([\Sigma_1]\times[\Sigma_1])\cap\ldots\ldots\cap([\Sigma_p]\times[\Sigma_p])\le\vartheta \quad (19)$$

The uniform convergence structure which represented as λ_0 is not only countable but also uniformly Hausdorff. Moreover, a filter *F* over $NL^{0}(\theta)$ converges to*v*is having elements $NL^{0}(\theta)$ w.r.t. λ_0 if and only if

$$\exists(\beta_n),(\omega_n)\subset\aleph L^{0}(\theta):$$
$$(1)n\in N\Rightarrow\beta_n\le\beta_{n+1}\le\omega_{n+1}\le\omega_n$$
$$(2)\sup\{\beta_n:n\in N\}=v=\inf\{\omega_n:n\in N\}$$
$$(3)[\{[\beta_n,\omega_n]:n\in N\}]\subseteq F. \quad (20)$$

The space $NL^{0}(\theta)$ completion w.r.t. the uniform convergence structure λ_0 can be demonstrated as the class of $NL(\theta)$, consisted with the suitable convergence structure. This above findings fulfils certainly as an application of the order completeness of $NL(\theta)$ as well as the property of approximation mentioned in eq. 18. The optimum uniform convergence outcome structure over $NL(\theta)$ is expressed as follows.

Definition 2: A filter T over $NL(\theta)\times NL(\theta)$ are lies in the class λ_0^{*} for some positive integer coefficients p, we find some results in such a way:

$$\forall k=1,2,3,\ldots\ldots\ldots p:$$
$$\exists(\beta_n^k),(\omega_n^k)\subset NL^{0}(\theta):$$
$$\exists v^k\in NL(\theta):$$
$$(1)\beta_n^k\le\beta_{n+1}^k\le\omega_{n+1}^k\le\omega_n^k,n\in N$$
$$(2)\sup\{\beta_n^k:n\in N\}=v^k=\inf\{\omega_n^k:n\in N\},$$
$$(3)\bigcap_{k=1}^{p}(([\Sigma^k]\times[\Sigma^k])\cap([v^k]\times[v^k]))\subseteq T. \quad (21)$$

3. Spaces of Generalized Function & its Solution

The space of generalized functions $NL^p(\theta)$ was shown in literature to attain results in the form of a huge family of systems of nonlinear partial differential equations. Although, as described in previous section, the existing solution are not sufficient enough to find initial as well as boundary values corresponding with a defined system of nonlinear differential equation. Furthermore,to add some more conditions, we required to construct the space $NL^p(\theta)$.

In the given reference, just focus on a system of g nonlinear differential equations:

$$G_u^p v(u,b) = H(u,b,.......,G_b^s G_u^r v_j(u,b),....) \tag{22}$$

With $u \in \Im$, $b \in \Im^{g-1}, p \geq 1, 0 \leq r \leq p, s \in \Re^{g-1}, |s| + r \leq p$ and with the Cauchy data

$$G_u^r v(u_0,b) = g_r(b), 0 \leq r \leq p, (u_0,b) \in H \tag{23}$$

on the hyperplane

$$H = \{(u_0,b) : b \in \Im^{g-1}\} \tag{24}$$

It is considered that equation 30 fulfil the criteria

$$\forall 0 \leq r \leq p : g_r \in \xi^{p-r} \Im^{g-1^g}\} \tag{25}$$

It eq. described above follows that the system of nonlinear partial differential equations shown in eq. 22 accepts a generalized solution in $NL^p(\Im^g)^g$. Although, this type of result may not be so easy to fulfil the initial condition (eq. 23) in a desirable situation. In order to inculcate the initial condition shown in eq. 23 into our findings, we explore spaces of functions mentioned below.

$$ML_g^p(\theta) = \left(v \in ML^p(\theta) \;\middle|\; \begin{array}{l} \forall j = 1,......g : \\ \forall 0 \leq r < p : \\ \forall s \in \Re^{g-1}, 0 \leq |s| + r \leq p : \\ (1) G_{bu}^{sp} v_j(b,u_0) = G^s g_{r,j}(b), b \in \Im^{g-1} \\ (2) G_{bu}^{sp} v_j ....Conti...at(b,u_0) \end{array} \right) \tag{26}$$

The partial differential operator is represented as $D_{j,u}^p$:

$$D_{j,u}^p : ML_g^p(\theta) \ni v \mapsto (Q \circ R)(D_u^p v_j) \in ML^0(\theta) \tag{27}$$

The way which achieve generalized outcomes of the initial value problem with the help of equations 22, 23 is necessary to use in the case of arbitrary systems of nonlinear differential equations. When we deal with the generalized results as an elements of the completion of the space $ML_g^p(\theta)$, it also equipped with a suitable uniform convergence structure. In this respect, we illustrate the following uniform convergence structure on $ML_{j,s,r}^0(\theta)$.

Definition 3: Let us consider that Σ contains of all nonempty order intervals in $ML_{j,s,r}^0(\theta)$. Also assume $F_{j,s,r}$ represent the class of filters on $ML_{j,s,r}^0(\theta) \times ML_{j,s,r}^0(\theta)$ that hold the following findings. There lies g∈ N in such a manner:

$$\begin{array}{l} \forall k = 1,2,3,.........p : \\ \exists \Sigma_k = (Q_n^k) \subseteq \Sigma : \\ \exists v_k \in NL(\theta) : \\ (1) Q_{n+1}^k \subseteq Q_n^k, n \in N \\ (2) \sup\{\inf Q_n^k : n \in N\} = v_k = \inf\{\sup Q_n^k : n \in N\} \\ (3)([\Sigma_1] \times [\Sigma_1]) \cap ....\cap ([\Sigma_p] \times [\Sigma_p]) \subseteq T \end{array} \tag{28}$$

Existence of Generalized Findings

A mapping associated with the system of nonlinear partial differential equations defined in equation PDEs (22) may be expressed in the following manner:

$$Z : ML_g^p(\theta) \to ML^0(\theta)^p \tag{29}$$

the components of which are defined through

$$Z_k : ML_g^p(\theta) \ni v \mapsto (Q \circ R)(G_{k,b}^n v + D_k(.,.,....,G_{bu}^{sr} v,...)) \in ML^0(\theta) \tag{30}$$

Thus, we reached on some conclusive remark of generalized outcome of the initial value problem shown in equations (22) & (23) in the regards of the space $ML_g^p(\theta)$ by suitably extending the mapping shown in eq.29 to a different mapping approach

$$Z^* : ML_g^p(\theta) \to ML^0(\theta)^p \tag{31}$$

So this type of extension is achieved through the uniform continuity of the mapping as in eq. 29. Thus, we have something shown below:

Theorem: The mapping shown in eq. 29 is said to be uniformly continuous.

Proof: A useful fundamental concept of the uniform convergence space $ML_{j,s,r}^0(\theta)$ and its completion $ML_{j,s,r}(\theta)$ relates to the inclusion mapping in such a manner:

$$j = ML_{j,s,r}^0(\theta) \to ML^0(\theta) \tag{32}$$

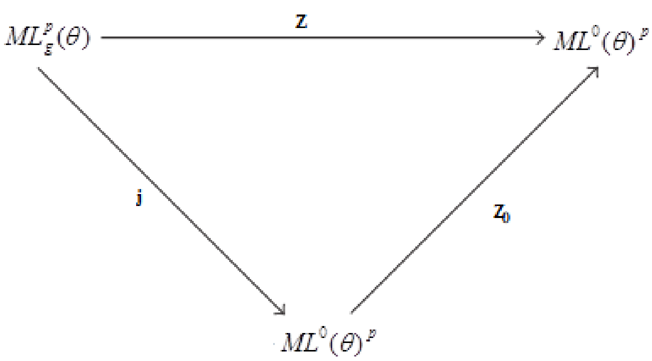
and its extension with the help of uniform continuity

$$j^* = ML_{j,s,r}(\theta) \to ML(\theta) \tag{33}$$

From equations 32 & 33 that the inclusion mapping

$$j : ML_g^p(\theta) \to ML^0(\theta)^p \tag{34}$$

is said to be uniform continuous & the solution can be demonstrate by the figure shown below:



and the uniform continuity of Z_0, which is the mapping demonstrated over $ML^0(\theta)^p$ by using the nonlinear partial differential operator (PDEs) as shown in literature [8].

Furthermore, as per theabove theorem the mapping in eq. 29 extends in a particular manner to a uniformly continuous mapping shown in eq. 31. Thus, the generalized initial value problem w.r.t. eq. 22 as well as eq. 23 is represented with the help of following equation

$$Z^* v^* = 0 \tag{35}$$

Where, the R.H.S. contains'0' represents the element in $ML(\theta)^p$ with every elements having zero value. A result of eq. 35 is called as a generalized outcome of eq. 21 & 22 based on the fundamental concept that all findings of eq. 35 fulfills the initial condition ina generalized way, as shown in below equation 36.

$$\forall k = 1,2,3,....,g$$
$$\forall 0 \le r \le p :$$
$$\forall s \in \Re^{g-1}, 0 \le r + |s| \le p :$$
$$G_{bu}^{sr^*} v^* = (u_0, b) = G^s g_{r,j}(u_0, b), b \in \Im^{g-1} \quad (36)$$

The extent to which the result explored in this paper may be represented as a classical solution over the domain of fundamental definition of the system of differential equations is unknown. Moreover, it can be easily said that, the result may be incorporated as the spaces of generalized functions which are nothing but the study of linear as well as nonlinear partial differential equations.

5. Conclusion

The proposed work demonstrate how these concepts defined in literature& further these results can be improved for initial as well as boundary value problems. However, the generalized findings of a huge family of nonlinear initial value problems are designed. It is clearly observed that the methods used to calculate or identify the existence of outcomes are similar to the techniques mentioned in [8] for the free problem. Moreover, we focus on the other finding of solving nonlinear partial differential equations w.r.t. the spaces of generalized functions. Finally, it can be said that functional analytic methods are not sufficient for finding solution for complicated initial as well as boundary value problems.

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