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Efficient numerical scheme for non-conservative form of a traffic flow model

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Abstract
This paper considers a macroscopic non-conservative form of traffic flow model known as Lighthill, Whitham and Richards (LWR) model appended with a linear velocity-density function. The model reads as a quasi-linear first order hyperbolic partial differential equation (PDE) and in order to incorporate initial and boundary data treated as an initial boundary value problem (IBVP). We presents the exact solution of the PDE as a Cauchy problem and the derivation of a finite difference scheme of non-conservative form of traffic flow model which is a first order explicit upwind difference scheme and establish well-posed-ness and stability condition for the scheme. Computer programs for the implementation of the numerical scheme and perform numerical experiments in order to verify some qualitative behaviour of traffic flow for various traffic parameters and relative errors and verify convergence of errors.

Keywords: Non-conservative traffic flow model, linear velocity-density function, numerical simulation

Introduction
The number of vehicles is increasing rapidly, in recent years; traffic congestion has become especially an acute problem. Therefore, an efficient traffic control and management is essential in order to grid of such huge traffic congestion. The aims of this analysis are principally represented by the maximization of vehicles flow, and the minimization of traffic congestions, accidents and pollutions etc. In this paper, we consider a macroscopic non-conservative form of traffic flow model first developed by Lighthill and Whitham (1955) and Richard (1956) shortly called LWR model based on [1, 4, 5]. As presented, we study finite difference method for first order non-linear PDE [6, 8, 9, 10] and based on these, we develop a finite difference scheme for a non-conservative form of traffic flow model as an (IBVP) which has been presented in numerical simulation. We develop computer programming for the implementation of the numerical scheme and perform numerical experiments in order to verify some qualitative behaviour of non-conservative form of traffic flow for various traffic parameters.

Mathematical Formulation of Non-Conservative Form of Traffic Flow Model
The traffic flow model is used to study traffic flow by collective variables such as traffic flow rate (flux) $q(x,t)$, traffic speed $v(x,t)$ and traffic density $\rho(x,t)$, all of which are functions of space $x \in R$ and time $t \in R^+$. The well-known LWR model [4, 5, 7] based on the principle of mass conservation reads as

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad (1)$$

Equation (1) is a first order partial differential equation called the equation of continuity with two unknowns $\rho(t,x)$ and $q(t,x)$ in one equation and is not solvable. Therefore, one needs to model the flux $q(t,x)$ as a function of $\rho(t,x)$ in order to obtain the equation (1) in a closed form. We consider velocity $v = v(\rho)$ as a function of density and therefore, we have the flux $q = q(\rho) = \rho v(\rho)$ (2)

Inserting a linear velocity-density closure relationship $v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$ and equation (2) takes the form $q = q(\rho) = \rho\, v(\rho) = \rho . v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) = v_{max}\left(\rho - \frac{\rho^2}{\rho_{max}}\right)$.

Therefore, equation (1) leads to formulate a non-linear first order hyperbolic partial differential equation (PDE) of the form

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad (3)$$

where $q'(\rho) = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)$. Equation (3) is known as Non-Conservative form of traffic flow model. The graph of the non-linear flux function given by equation (2) is known as Fundamental Diagram as sketched below.

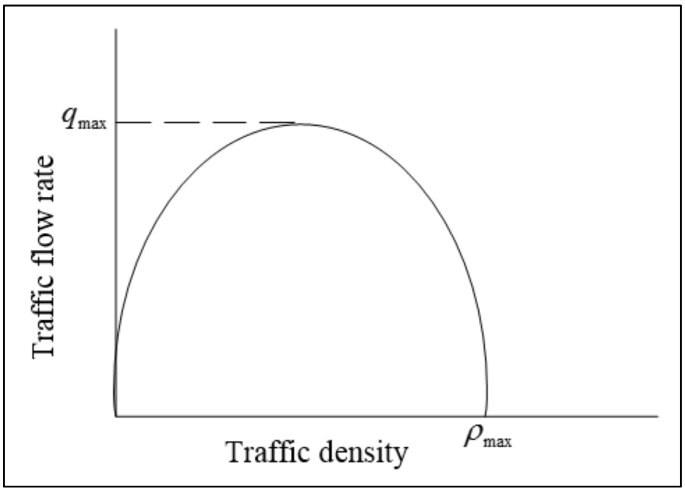

Fig 1: Fundamental Diagram of Traffic Flow

Qualitative Behavior of Linear Velocity Function $v = v(\rho)$

There is a connection between traffic density and vehicle velocity. If there is more vehicles are on a road then their velocity will be slower. On the basis of observations of traffic flow, we make a basic simplifying assumption that the velocity of a car at any point along the highway depends only on the traffic density. Drivers speed up when traffic is sparse and they slow down when traffic is dense. Thus there is a direct relationship between traffic density and traffic velocity as $q = \rho v$. Now in order to deal with the non-linear model (1) it is necessary to understand the relation $v = v(\rho)$ a bit further. Based upon the intuition mentioned above, one may assume that a driver will drive fastest, with velocity, say v_{max}, when the density is at its smallest value, $\rho_{min} \to 0$. The velocity decreases as the density increases, which is a statement about the slope of the v verse ρ curve. Assume further that the traffic is bumper-to bumper, i.e., $v = 0$, at some maximum density ρ_{max} with $\rho_{max} < \frac{1}{L}$, where L is the average length of a vehicle. We summarize these experience-born intuitions in mathematical requirements on the function, $v(\rho)$:

$$\left.\begin{aligned} & v(\rho = 0) = v_{max}, \\ & \frac{dv}{d\rho} \le 0, \\ & v(\rho = \rho_{max}) = 0. \end{aligned}\right\} \qquad (4)$$

Qualitative Behavior of Linear Traffic Flow (flux) $q = q(\rho)$

The flow or flux given by $q = v_{max}\left(\rho - \frac{\rho^2}{\rho_{max}}\right)$ is a linear function. The maximum flow (flux) occurs when its slope vanishes and $\frac{d^2 q}{d\rho^2} < 0$ i.e. $\frac{d^2 q}{d\rho^2}$ is negative. Now,

$$\frac{dq}{d\rho} = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) = 0$$

$$\Rightarrow \left(1 - \frac{2\rho}{\rho_{max}}\right) = 0$$

$$\Rightarrow 2\rho = \rho_{max}$$

$$\therefore \rho = \frac{\rho_{max}}{2}$$

Therefore, $\frac{d^2 q}{d\rho^2} = -\frac{2v_{max}}{\rho_{max}} < 0,\ \rho \in (\rho_{min}, \rho_{max})$. so, $\frac{d^2 q}{d\rho^2}$ is negative. Therefore, $q(\rho)$ is convex and the flow (flux) is maximum at $\rho = \frac{\rho_{max}}{2}$ and the maximum flow (flux) is

$$q_{max} = v_{max}\left(\frac{\rho_{max}}{2} - \frac{\left(\frac{\rho_{max}}{2}\right)^2}{\rho_{max}}\right)$$

$$\Rightarrow q_{max} = v_{max}\left(\frac{\rho_{max}}{2} - \frac{\rho_{max}}{4}\right)$$

$$\therefore q_{max} = \frac{1}{4}\rho_{max} v_{max}$$

Exact Solution of Non-Conservative Form of Traffic Flow Model by the Method of Characteristics

The non-linear PDE of a non-conservative form of traffic flow model (3) can be solved if we know the traffic density at a given initial time, i.e. if we know the traffic density at a given time t_0 we can predict the traffic density for all time $t \geq t_0$, in principle. Then we have to solve an initial value problem (IVP) of the form

$$\left.\begin{aligned} &\frac{\partial\rho}{\partial t} + q'(\rho)\frac{\partial\rho}{\partial x} = 0;\ q'(\rho) = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right) \\ &\text{with}\quad \rho(t_0, x) = \rho_0(x) \end{aligned}\right\} \tag{5}$$

The IVP (5) can be solved by the method of characteristics as follows. PDE in (5) can be written as

$$\frac{\partial\rho}{\partial t} + q'(\rho)\frac{\partial\rho}{\partial x} = 0\ ;\text{ where }\ q'(\rho) = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)\frac{\partial\rho}{\partial x} = 0$$

We have $\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} \Rightarrow \frac{d\rho}{dt} = 0$ (6)

where $\frac{dx}{dt} = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)$ (7)

From equation (7), we have $x(t) = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)t + x_0$ is called the characteristics curve starting from $x = x_0$. Equation (5) yields that along the characteristics curve the density $\rho(t, x)$ is constant, equating its value in $t = 0$,

$$\rho(t,x)=\rho(x_0,0)=\rho_0(x_0)=\rho_0\left(x-v_{max}\left(1-\frac{2\rho}{\rho_{max}}\right)t\right) \tag{8}$$

which is the exact solution of the IVP (5) and is in implicit form. However, in reality it is very difficult to approximate the initial density $\rho_0(x)$ from given initial data, it may cause a huge error. Therefore, there is a demand of some efficient numerical methods for solving the IVP (5) as an initial boundary value problem (IBVP), which is also appended with boundary condition.

Finite Difference Scheme for Numerical Solution of Non-Conservative Form of Traffic Flow Model

In this section we present the derivation of finite difference scheme for our model with linear velocity-density relation appended with initial and boundary condition. We consider our specific non-linear PDE of non-conservative form of traffic flow model as an IBVP:

$$\left.\begin{aligned}&\frac{\partial\rho}{\partial t}+q'(\rho)\frac{\partial\rho}{\partial x}=0,\ t_o\le t\le T,\ a\le x\le b\\ &\text{with i.c. }\rho(t_o,x)=\rho_o(x);\ a\le x\le b\\ &\text{and b.c. }\rho(t,a)=\rho_a(t);\ t_o\le t\le T,\end{aligned}\right\} \tag{9}$$

where $q'(\rho)=v_{max}\left(1-\dfrac{2\rho}{\rho_{max}}\right)$.

In order to develop the scheme, we discretize the space and time. The discretization of time derivative $\frac{\partial\rho}{\partial t}$ in the IBVP (9) at any discrete point (t_n,x_i) for $i=1,\ldots\ldots,M$, $n=0,\ldots\ldots.,N-1$; by the first order forward difference formula $\frac{\partial\rho(t_n,x_i)}{\partial t}\approx\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}$ and the discretization of space derivative $\frac{\partial\rho}{\partial x}$ by first order backward difference formula $\frac{\partial\rho(t_n,x_i)}{\partial x}\approx\frac{\rho_i^n-\rho_{i-1}^n}{\Delta x}$. Now equation (9) takes the form

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+q'(\rho_i^n)\frac{\rho_i^n-\rho_{i-1}^n}{\Delta x}=0$$

$$\Rightarrow\ \rho_i^{n+1}=\rho_i^n-q'(\rho_i^n)\frac{\Delta t}{\Delta x}\left(\rho_i^n-\rho_{i-1}^n\right) \tag{10}$$

where $q'(\rho_i^n)=v_{max}\left(1-\dfrac{2\rho_i^n}{\rho_{max}}\right)$. This equation is known as explicit upwind difference scheme in non-conservative traffic flow model. In the finite difference scheme, the initial data ρ_i^0 for all $i=1,\ldots\ldots,M$; is the discrete versions of the given initial value $\rho_0(x)$ and the boundary data ρ_a^n for all $n=0,\ldots\ldots.,N-1$ are the discrete versions of the given boundary value $\rho_a(x)$.

Stability and Physical constraints condition

In explicit upwind difference scheme for non-linear PDE of traffic flow maximum velocity is unknown but fortunately it is known in our specific model by the velocity- density relationship $q\left(\rho_i^n\right)=v_{max}\left(\rho_i^n-\dfrac{(\rho_i^n)^2}{\rho_{max}}\right)$ i.e. $q'(\rho_i^n)=v_{max}\left(1-\dfrac{2\rho_i^n}{\rho_{max}}\right)\ge 0$

$$\Rightarrow\ \rho_{max}\ge 2\rho_i^n\ \text{ i.e. }q'(\rho_i^n)\le v_{max} \tag{11}$$

The explicit finite difference scheme (10) takes the form

$$\rho_i^{n+1}=\rho_i^n-q'(\rho_i^n)\frac{\Delta t}{\Delta x}\left(\rho_i^n-\rho_{i-1}^n\right)$$

$$\Rightarrow\ \rho_i^{n+1}=(1-\lambda)\rho_i^n+\lambda\rho_{i-1}^n \tag{12}$$

where $\lambda := q'(\rho_i^n)\dfrac{\Delta t}{\Delta x}$. The equation (12) implies that if $\lambda \le 1$, the new solution is a convex combination of the two previous solutions. That is the solution at new time-step $(n+1)$ at a spatial node is an average of the solutions at the previous time-step at the spatial nodes i and $i-1$. This means that the extreme value of the new solution is the average of the extreme values of the previous two solutions at the two consecutive nodes. Therefore, the new solution continuously depends on the initial value ρ_i^o, $i = 1, 2, 3, \ldots\ldots, M$ and the explicit finite difference scheme is stable for $\lambda := q'(\rho_i^n)\dfrac{\Delta t}{\Delta x} \le 1$ and then by condition (11) implies that $\lambda := \dfrac{v_{max}\,\Delta t}{\Delta x} \le 1$ (13)

This is the stability condition. Thus whenever one employs the stability condition $\lambda := v_{max}\dfrac{\Delta t}{\Delta x} \le 1$, the well-posed-ness condition (11) can be guaranteed immediately by choosing $\rho_{max} = k \max_i \rho_o(x_i)$, $k \ge 2$.

Numerical Experiments and Results Discussion

We implement the EUDS by developing a computer programming code and perform numerical simulation as described below.

Now we consider the initial density using sine function and the constant one sided boundary value for EUDS is $\rho(t,0) = 21/0.1\,km$ to perform numerical computation in the spatial domain [0, 10] in km. For the above initial and boundary conditions with $v_{max} = 60.12\,km/hour = 0.167(0.1\,km/sec)$; satisfying the physical constraint condition (13); $\rho_{max} = 5\max_i \rho_o(x_i) = 550$ vehicles/km in the spatial domain [0 km, 10 km] we perform the numerical experiment for 6 minutes in Δt = 0.01 time steps for a highway of 10 km in 401 spatial grid points with step size Δx = 100 meters = 0.25 which guarantees the stability condition (13); $\gamma = 0.00668 < 1$.

Figure-2 shows the initial density and the density after six minutes and also combined. Figure-3(a) and Figure-3(b) shows the density $\rho(t,x)$ profiles and velocity $v(t,x)$ profiles 1st, 2nd and 3rd minutes respectively and one can observe from the two figures that the density and velocity are maintaining the negative relation, as given by $v(\rho) = v_{max}\left(1 - \dfrac{\rho}{\rho_{max}}\right)$, throughout the computational process as expected and also Figure-3(c) show the flux $q(t,x)$ profiles for first three different times of EUDS for non-conservative form of traffic flow model. Figure-4 shows the compare profile of density, velocity and flux in 1st three minutes.

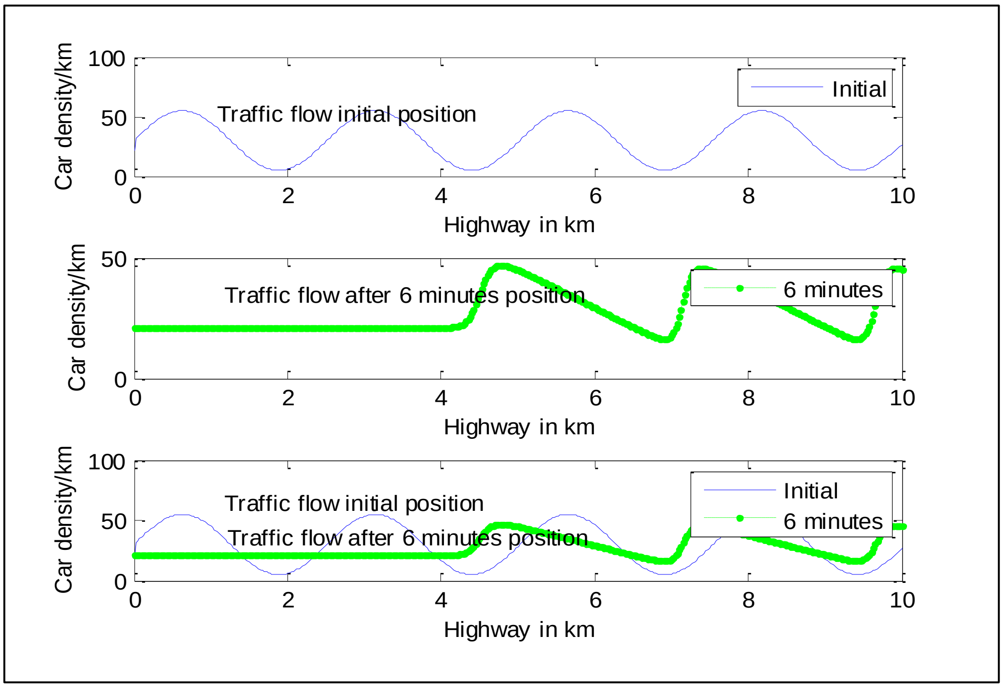

Fig 2: Initial density and the density after six minutes and also combined in a 10 km highway

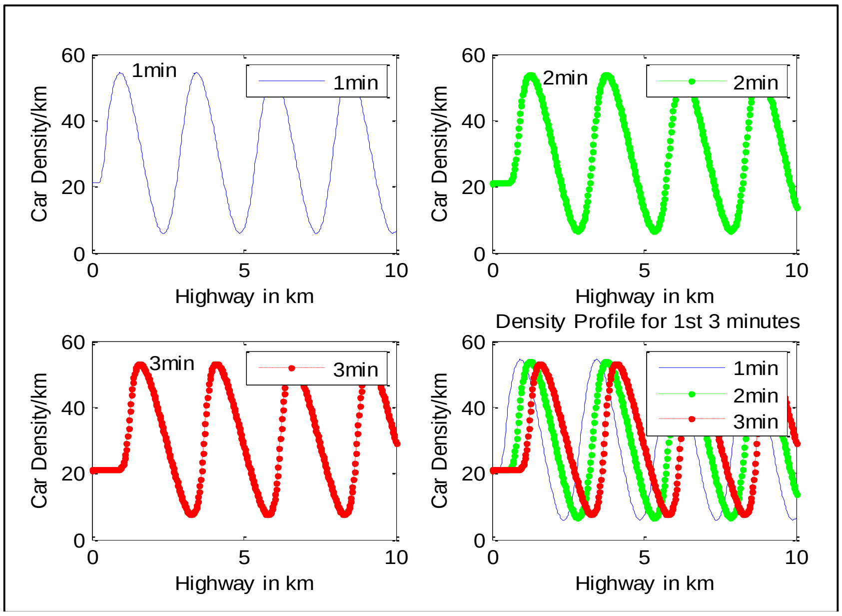

Fig 3(a): Density profile of first three minutes in a 10 km highway

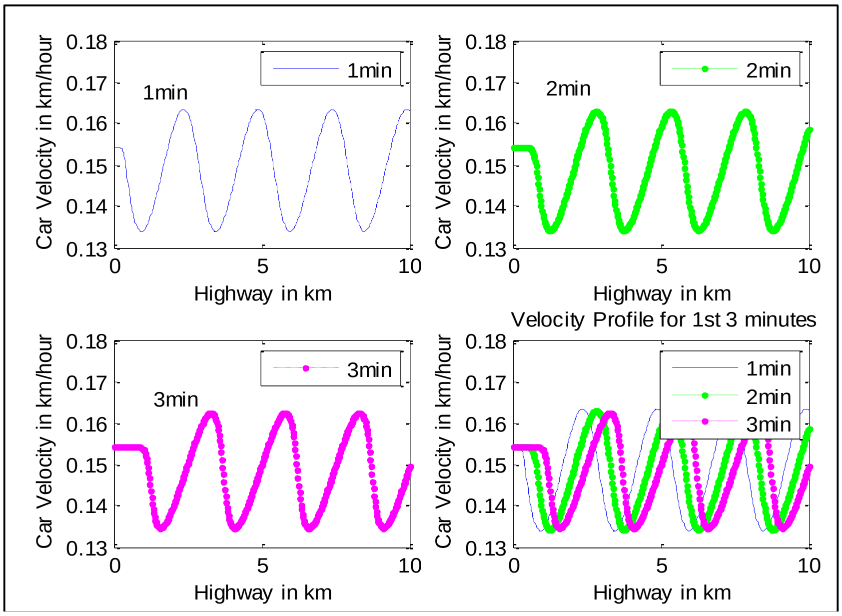

Fig 3(b): Velocity profile of first three minutes in a 10 km highway

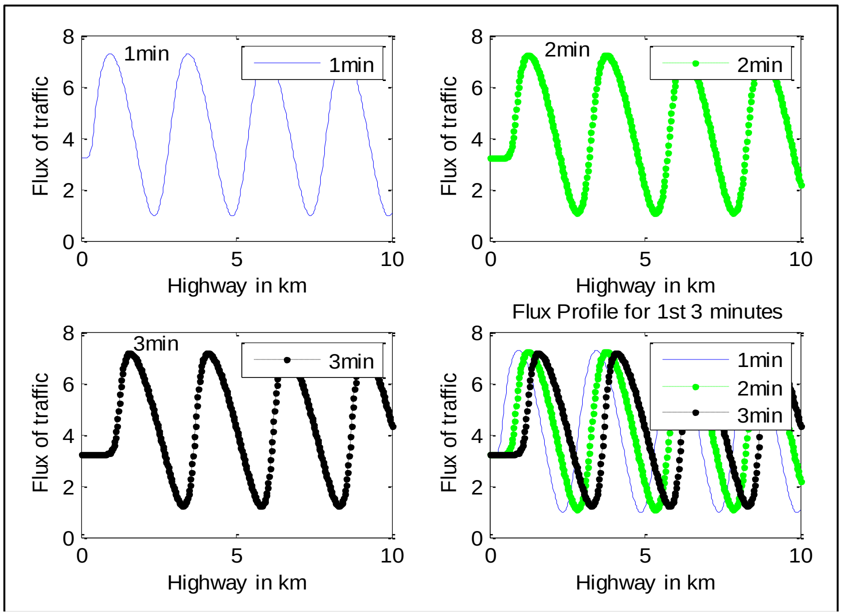

Fig 3(c): Flux profiles of first three minutes in 10 km highway

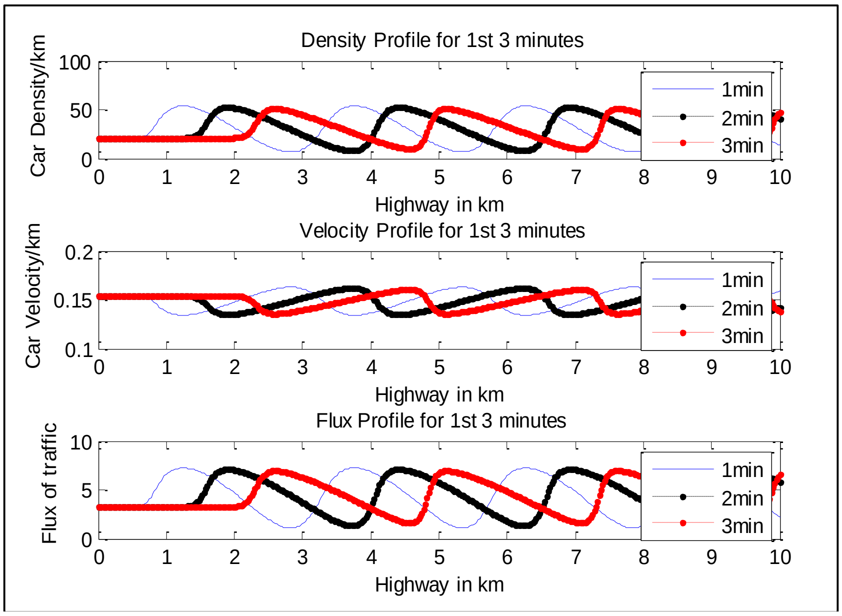

Fig 4: Compare profile of traffic density, velocity and flux of first three minutes in a 10 km highway

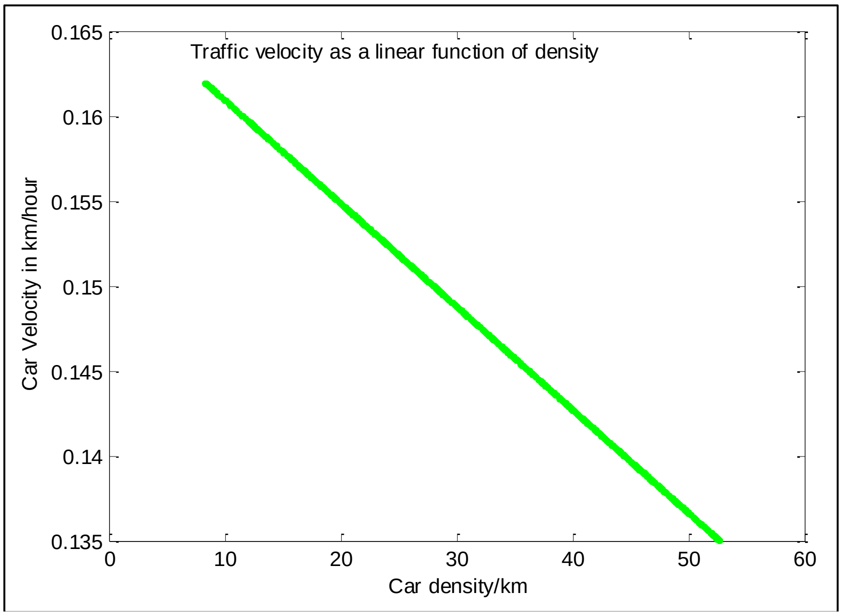

Fig 5: Traffic velocity as a function of density

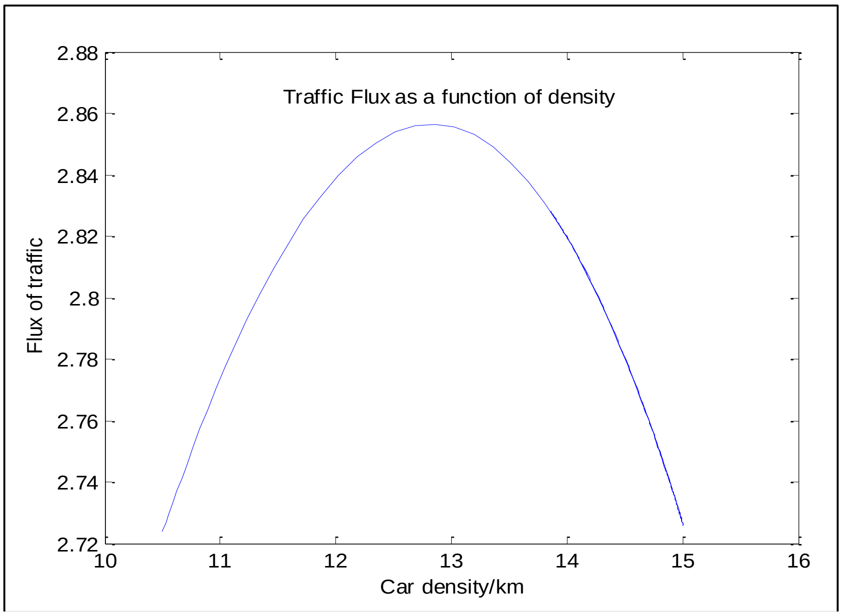

Fig 6: Traffic Flux as a function of density

Finally, Figure-5 presents the computed car velocity as a function of density and Figure-6 shows the computed traffic flow (flux) as a function of density. Figure-6 same as the qualitative behavior, the well-known fundamental diagram as figure-1.

Error Estimation of Numerical Scheme

In order to perform error estimation for density (ρ), we consider exact solution (8) with initial condition i.e. linear function $\rho_o(x) = \frac{1}{2}x$, we have

$$\rho(t,x) = \rho_0(x_0) = \frac{1}{2}\left(x - v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)t \right)$$

$$\Rightarrow \rho(t,x) = \frac{(x - v_{max}t)/2}{(1 - v_{max}t)/\rho_{max}}.$$

We prescribe the corresponding boundary value for EUDS by the equation

$$\rho_a(t) = \rho(t,x_a) = \frac{(x_a - v_{max}t)/2}{(1 - v_{max}t)/\rho_{max}}$$

We compute the relative error in L_1-norm defined by $\|e\|_1 = \frac{\|\rho_e - \rho_n\|_1}{\|\rho_e\|_1}$ for all time ρ_e is the exact solution and ρ_n is the numerical solution computed by finite difference scheme.

Figure-7 shows the comparison of the exact solution and numerical solution in the bound (t,x) plane and $t = [0,\ 6\,min], x = [0, 10\,km]$. Figure-8 below shows the relative error for density (ρ) of explicit upwind difference scheme, which remains 0.00012 which is quite acceptable. Figure-9 presents that the density (ρ) error is decreasing with respect to the smaller descretization parameters Δt and Δx which shows the convergence of explicit upwind difference scheme.

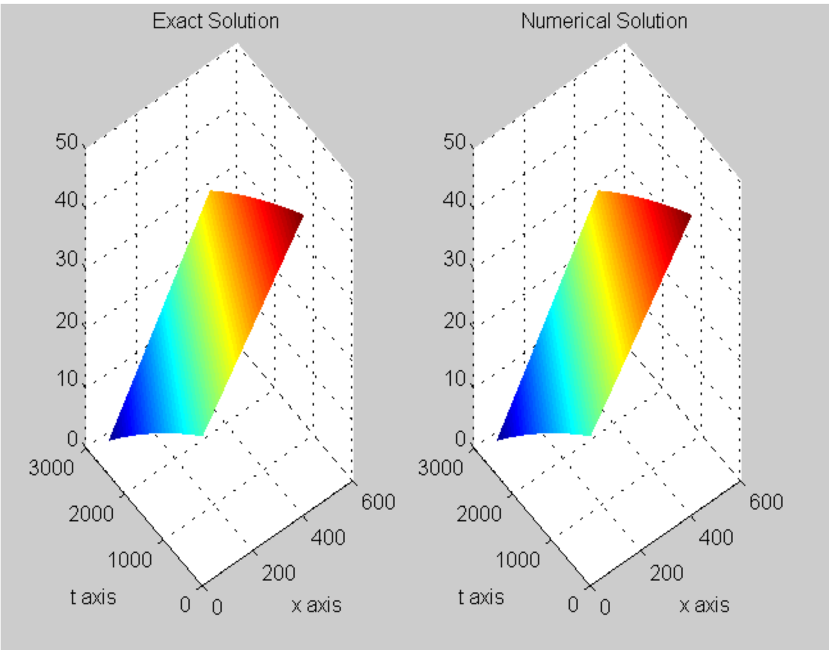

Fig 7: Compare error between exact solution and numerical solution

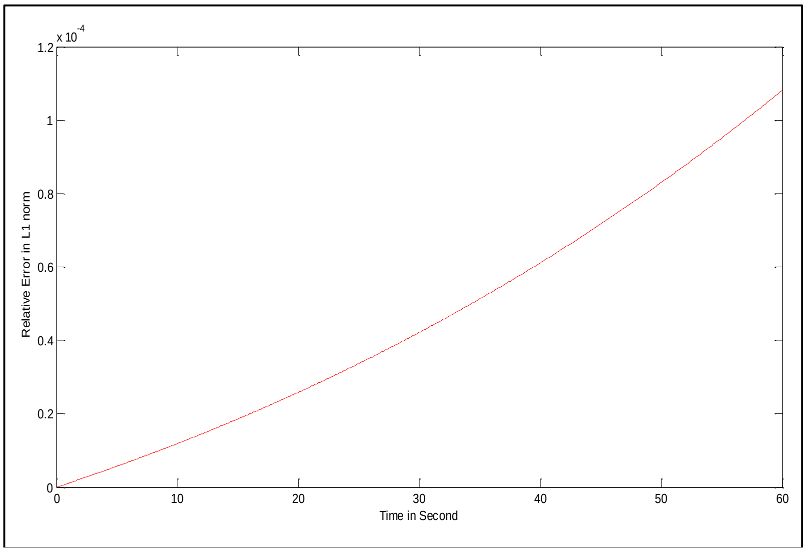

Fig 8: Relative Errors

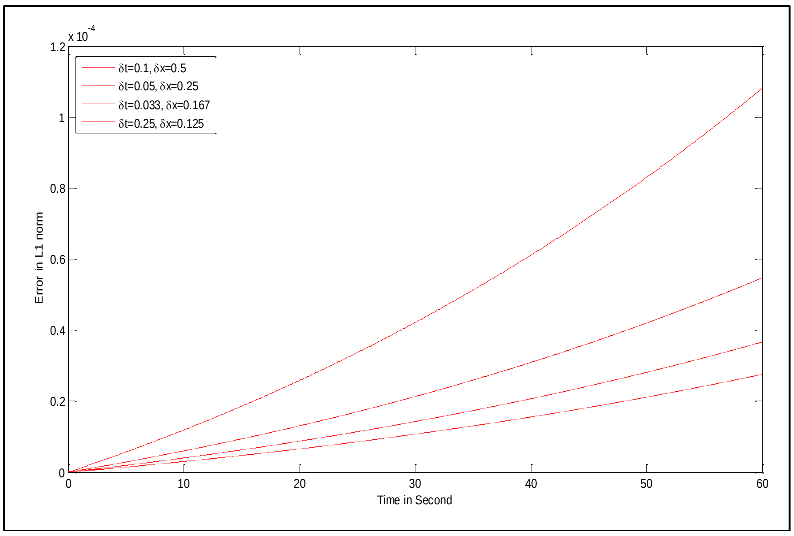

Fig 9: Convergence of Errors

Conclusion

The computational result obtained by implementing the analogous version of non-conservative form of traffic flow model of the numerical solution by EUDS shows the accuracy up to four decimal places and a good rate of convergence. Performing numerical simulation, we have verified some qualitative traffic flow behavior for various traffic parameters. Finally, we have presented diagram of traffic flow using this scheme, which is a very good qualitative agreement for traffic flow model. In our model, we have considered only single lane highway. The model can be extended for multi-lane traffic flow model which we leave as our future work.

References

1. Bretti G, Natalini R, Piccoli B. A Fluid-Dynamic Traffic Model on Road Networks, Comput Methods Eng., CIMNE, Barcelona, Spain. 2007; 14:139-172.
2. Andallah LS, Ali S, Gani MO, Pandit MK, Akhter J, A Finite Difference Scheme for a Traffic Flow Model Based on a Linear Velocity-Density Function. Jahangirnagar University Journal of Science. 2009; 32:61-71.
3. Kuhne R, Michalopoulos P. Continuum Flow Models, 1997.
4. Haberman R. Mathematical Models. Prentice-Hall, Inc., Delhi, 1977.
5. Klar A, Kuhne RD, Wegener R. Mathematical Models for Vehicular Traffic. Technical University of Kaiserslautern, Dept. of Math., TU of Kaiserlautern, Germany, 1996.
6. Zhang HM. A finite difference approximation of non-equilibrium traffic flow model, Transportation Research Part-B: Methodological (Elsevier), 2001; 35(4):337-365.
7. Dym CL. Principles of Mathematical Modeling, Academic press, 2004.
8. Larsson S, Thomee V. Partial Differential Equations with Numerical Methods, Second Printing Springer-Verlag Berlin Heidelberg, 2005,
9. Leveque RJ. Numerical Methods for Conservation Laws. 2nd Edition, Springer, Berlin, 1992
10. Daganzo CF. A Finite Difference Approximation of the Kinematic Wave Model of Traffic Flow. Transportation Research Part B: Methodological. 1995; 29:261-276.