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Russell measures in data envelopment analysis

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Abstract
A producer who cannot vary his output enquires for possible reduction of inputs. If reduction is not possible he is efficient, otherwise inefficient. In this case the producer minimizes the total cost of production. Alternatively, if inputs cannot be varied, which is often in short run, the entrepreneur enquires for further output augmentation, if such augmentation is not possible he is efficient, otherwise inefficient. In this situation the implicit assumption is revenue maximization. However, in long run, inputs as well as outputs can be varied simultaneously, where the underlying optimization is profit maximization. If the producer neither reduces inputs nor augments outputs, production is profit efficient, otherwise inefficient.

Keywords: Russell measures, data envelopment analysis

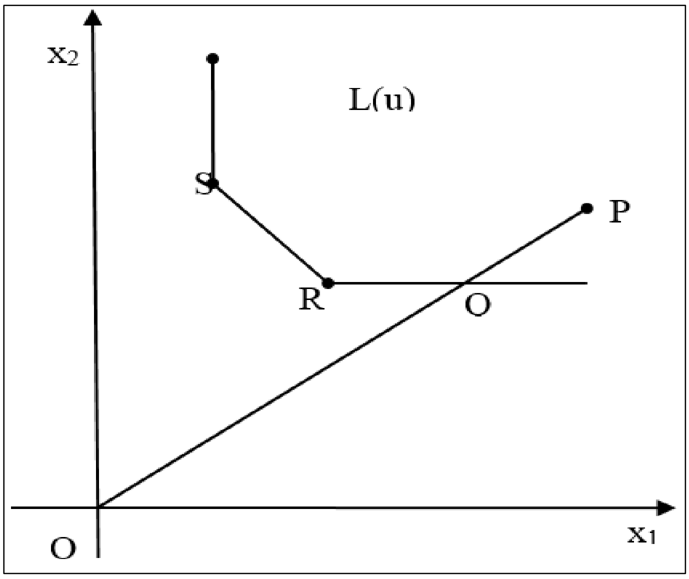
1. Introduction

The theoretical literature on productive efficiency originated with the work of The first attempt to estimate efficiency was found in used linear programming techniques to estimate efficiency in U.S. agriculture. Research on efficiency estimation continued on through the development of Stochastic Frontier Analysis (SFA), (Aigner, Lovell, and Schmidt 1977) [3]. DEA was developed at about the sametime by Each of these techniques has subsequently been extended and developed further (see Kumbhakar and Lovell 2000 for a modern textbook treatment of SFA and Cooper, Seiford, and Zhu 2004 for DEA). The empirical application of these techniques to estimate efficiency and/or productivity has resulted in a multitude of studies in numerous areas. Researchers have employed SFA or DEA (and in some cases both techniques) to analysis of agriculture, utilities, education, health economics, transportation, labour economics, management science, environmental economics, and financial institutions. As an example of the amount of output, present a survey of research on the efficiency of financial institutions with a count of 130 studies across 21 countries.

2. Russell Measure of Input Technical Efficiency

In a k input, m output technology, Russell's measure of input technical efficiency can be viewed as a linear programming problem.

$$R_i(u,x) = Min\left\{ \frac{\sum_{i=1}^{k} \lambda_i}{k} : \left(\lambda_1 x_1, \lambda_2 x_2, \ldots\ldots\ldots \lambda_k x_k\right) \in L(u) \right\} \quad \ldots. (2.1)$$



- P is inefficient DMU
- Isoquant L(u) = Eff L(u) U Weff L(u)
- Radial measure of technical efficiency: $\frac{OQ}{OP}$
- Q is weakly efficient input combination
- Input x_1 can be further reduced, so that P can be compared with R that belongs to Eff L (u)
- The line segment SR constitutes the efficient subset of Isoq L (u)
- The Russell measure of technical efficiency minimizes $\frac{\lambda_1 + \lambda_2}{2}$,

Subject to the condition that

$$0 \le \lambda_1 \le 1, \quad 0 \le \lambda_2 \le 1, \qquad (\lambda_1 x, \lambda_2 x_2) \in L(u)$$

To estimate Russell measure of technical efficiency, we solve the Following linear programming problem

$$\text{Min} \sum_{k=1}^{s} \lambda_k / s$$

Subject to

$$\left.\begin{array}{ll} \sum_{j=1}^{n} \delta_j u_{ij} \ge u_{i0}, & i = 1,2,\ldots\ldots.m \\ \sum_{j=1}^{n} \delta_j x_{lj} \ge \lambda_i x_{l0}, & l = 1,2,\ldots\ldots.s \end{array}\right\} \quad \ldots.\ (2.2)$$

$$\delta_j \ge 0$$

$$0 \le \lambda_2 \le 1$$

Minimization of $\frac{\sum_{k=1}^{s} \lambda_k}{s}$ is same as minimization of $\sum_{k=1}^{s} \lambda_k$

3. Russell Cost Measure of Technical Efficiency

The Farrell input technical efficiency can be given a Cost interpretation.

$$ITE = \frac{p(\lambda x_0)}{px_0} \qquad \text{…. (3.1)}$$

- λx_0 is radially reduced input that belongs to isoquant of the input level set $L(u_0)$
- $p(\lambda x_0)$ is cost of λx_0
- px_0 is cost of x_0
- $ITE = \lambda$
- cost interpreted ratio also leads to the same input technical efficiency measure

However the cost interpretation given to Russell input technical efficiency does not yield input technical efficiency measure. This phenomenon occurs since Russell technical efficiency measure is non-radial.

Let λ_1 and λ_2 be the rates at which inputs are reduced so that $(\lambda_1 x_{10}, \lambda_2 x_{20}) \in \text{Eff } L(u)$

- (x_{10}, x_{20})Is observed input vector.
- (p_1, p_2) is input price vector
- cost of observed input vector: $p_1 x_{10} + p_2 x_{20}$
- cost of Russell technically efficient input vector: $\lambda_1 p_1 x_{10} + \lambda_2 p_2 x_{20}$
- Russell cost measure of input technical efficiency is,

$$\frac{\lambda_1 p_1 x_{10} + \lambda_2 p_2 x_{20}}{p_1 x_{10} + p_2 x_{20}}$$

Where the denominator is observed cost.

More generally, fork input case, this measure takes the form,

$$\frac{\sum_{i=1}^{k} \lambda_i p_i x_{i0}}{\sum_{i=1}^{k} p_i x_{i0}} \qquad \text{…. (3.2)}$$

4. Russell Measure of Input Allocative Efficiency

The Russell measure of input allocative efficiency is defined as the ratio of factor minimal cost to the cost measure of Russell input technical efficiency.

$$\frac{Q(u_0, p)}{\sum_{i=1}^{k} \lambda_i p_i x_{i0}} \qquad \text{…. (4.1)}$$

Where $Q(u_0, p)$ is the factor minimal cost obtained by solving the following optimization problem:

$$Q(u_0, p) = \underset{x}{Min}\{px : x \in L(u_0)\}$$

5. Russell Measure of Overall Technical Efficiency

The Russell measure of overall technical efficiency depends on the input and output vectors, viz., x_0 and u_0 of the decision making unit in focus. Consequently, the measure is defined as,

$$\frac{Q(u_0, p)}{px_0}$$

However, the above ratio measures overall input cost efficiency in Farrell's measurement.

$$\frac{Q(u_0, p)}{px_0} = \left(\frac{Q(u_0, p)}{\sum_{i=1}^{k} \lambda_i p_i x_{i0}} \right) \left(\frac{\sum_{i=1}^{k} \lambda_i p_i x_{i0}}{px_0} \right) \quad \text{…. (5.1)}$$

Thus, Russell measure of overall input technical efficiency can be decomposed into the product of Russell input allocative efficiency and measure of overall technical efficiency.

6. Russell Output Measures of Productive Efficiencies

- Efficiencies are measured comparing the observed output vectors with efficient output vectors which belong to Eff P(x_0)
- P(x) is output level set
- Augmentation of additional outputs are non-radial
- The various efficiencies are obtained by appropriately formulating and solving linear programming problems.

i) Russell Output Measure of Technical Efficiency

Let x_0 be the input vector of the DMU whose output efficiency is under evaluation. The Russell output measure of technical efficiency is defined as,

$$R_0(x_0, u_0) = \mathrm{Max}\left\{ \frac{\sum_{i=1}^{m} \theta_i}{m} : (\theta_1 u_1, \ldots.\theta_m u_m) \in p(x_0), \theta_i \geq 1, i = 1,2,\ldots m \right\} \quad \text{…. (6.1)}$$

Maximizing $\dfrac{\sum_{i=1}^{m} \theta_i}{m}$ is same as maximizing the sum $\sum_{i=1}^{m} \theta_i$

$$mR_0(x_0, u_0) = Max\left\{ \sum_{i=1}^{m} \theta_i : (\theta_1 u_1, \theta_2 u_2, \ldots\ldots\ldots\ldots.\theta_m u_m) \in p(x_0) \right\}$$

ii) Russell Revenue Measure of Output Technical Efficiency

Let u_0 be the output vector of the DMU whose efficiency is under evaluation. If r is the revenue vector, the Russell revenue measure of output technical efficiency is the ratio of potential output revenue to the realized revenue at output price vector r. The measure takes the following form:

$$\frac{\sum_{j=1}^{m} \theta_j r_j u_{0j}}{\sum_{j=1}^{m} r_j u_{0j}} \quad \text{…. (6.2)}$$

The above measure reduced to the Farrell output technical efficiency measure if and only if,

$$\theta_j = \theta, j = 1,2,\ldots\ldots m$$

Russell revenue measure of output technical efficiency may be viewed as a derived measure.
If production technology is piecewise linear, we first solve the following linear programming problem:

$$Max\sum_{j=1}^{m}\theta_j$$

$$\left.\begin{aligned}&\text{Subject to } \sum_{i=1}^{n}\lambda_i x_{ij} \le x_{0j}, \qquad j=1,2,\ldots\ldots m\\ &\sum_{i=1}^{n}\lambda_i u_{ij} \ge \theta_j u_{0j}, \qquad j=1,2,\ldots\ldots m\end{aligned}\right\} \quad \ldots.(6.3)$$

$$\lambda_i \ge 0$$

Since r_j, u_{0j} are known and θ_j are obtained as solution of the above LPP we obtain,

$$\sum_{j=1}^{m}\theta_j r_j u_{0j}$$

iii) Russell Overall Output Efficiency Measure

The overall output efficiency measure of Russell is the ratio of maximal revenue to observed revenue. It depends on output price vector, observed input and output vectors of the DMU whose efficiency is under evaluation.

Maximum revenue can be estimated solving the following optimization problem:

$$R(x_0, r) = \max_u \{ru : u \in p(x_0)\}$$

If production technology is piecewise linear, under certain assumptions potential revenue can be obtained by solving the following linear programming problem.

$$R(x_{0,} r) = \max \quad ru$$

$$\left.\begin{aligned}&\text{Subject to } \sum_{i=1}^{n}\lambda_i x_i \le x_0\\ &\sum_{i=1}^{n}\lambda_i u_i \ge u\\ &\lambda_i \ge 0\end{aligned}\right\} \quad \ldots.(6.4)$$

7. Russell Output Allocative Efficiency Measure

This measure is viewed as the ratio of potential revenue to the revenue attained at

$$(\theta_1 u_1, \theta_2 u_2, \ldots\ldots\ldots\ldots\theta_m u_m) \in Eff\ p(x_0)$$

- Potential revenue: $R(x_{0,} r)$
- $(\theta_1 u_1, \theta_2 u_2, \ldots\ldots\ldots\ldots\theta_m u_m) \in Eff\ p(x_0)$ and

The revenue attained at this point is

$$\sum_{j=1}^{m}\theta_j r_j u_{0j}$$

- Allocative efficiency measure:

$$\frac{R(x_0, r)}{\sum_{j=1}^{m} \theta_j r_j u_{0j}}$$

Russell overall output efficiency measure can be decomposed into the product of Russell revenue measure of output technical efficiency and Russell output allocative efficiency measure.

$$\frac{R(x_0, r)}{ru_0} = \left[\frac{R(x_{0,} r)}{\sum_{j=1}^{n} \theta_j r_j u_{0j}}\right]\left[\frac{\sum_{j=1}^{m} \theta_j r_j u_{0j}}{ru_0}\right] \quad \text{…. (7.1)}$$

8. Russell Measures-Scale Efficiency

i) To find Russell measure of pure technical efficiency we solve the following problem.

$$R_I(u_0, x_0) = \underset{\lambda}{\mathrm{Min}}\left\{\frac{\sum_{i=1}^{s}\lambda_i}{s} : \ (\lambda_1 x_{01,} \lambda_2 x_{02,} \ldots\ldots_s x_{0s,}) \in L^v(u_0)\right\} \quad \text{… (8.1)}$$

Where $L^v(u_0)$ is the input level set consistent with variable return to scale. For linear production technology we solve the following:

$$KR_I^v(u_0, x_0) = \min \sum_{i=1}^{s} \lambda_i$$

Subject to

$$\left.\begin{array}{l} \sum_{j=1}^{n} \delta_j x_j \le x_0^1 \\ \\ \sum_{j=1}^{n} \delta_j u_j \ge u_0 \end{array}\right\} \quad \text{…. (8.2)}$$

$$\delta_j \ge 0$$

$$\sum_{j=1}^{n} \delta_j = 1$$

Where $x_0^1 = (\lambda_1 x_{01,} \lambda_2 x_{02,} \ldots\ldots.\lambda_s x_{0s,})^T$

ii) Russell measure of technical efficiency can be obtained solving the optimization problem:

$$R_I^k(u_0, x_0) = \underset{\lambda}{\mathrm{Min}}\left\{\frac{\sum_{i=1}^{s}\lambda_i}{s} : \ x^{1`}_0 \in L^k(u_0)\right\}$$

Where $L^k(u_0)$ is the input level set consistent with constant returns to scale. If production technology is piecewise linear,

$$KR_I^k(u_0, x_0) = \min \sum_{i=1}^{s} \lambda_i$$

Such that

$$\left.\begin{aligned} &\sum_{j=1}^{n} \delta_j x_j \leq x_0^1 \\ &\sum_{j=1}^{n} \delta_j u_j \geq u_0 \\ &\delta_j \geq 0 \\ &\sum_{j=1}^{n} \delta_j = 1 \end{aligned}\right\} \quad \text{…. (8.3)}$$

Where $x_0^1 = (\lambda_1 x_{01,} \lambda_2 x_{02,} .....\lambda_k x_{0k,})^T$

Russell input scale efficiency is defined as,

$$RISE = \frac{R_I^k(u_0, x_0)}{R_I^v(u_0, x_0)}$$

- $L^k(u_0, x_0) \supseteq L^v(u_0, x_0)$
- $R_I^k(u_0, x_0) \leq R_I^v(u_0, x_0)$
- $0 \leq \mathrm{RI\,SE}(u_0, x_0) \leq 1$

iii) Russell output pure technical efficiency can be found solving the following optimization problem.

$$R_0^v(x_0, u_0) = Max\left\{\frac{\sum_{i=1}^{m} \theta_i}{m} : (\theta_1 u_{01}, ...\theta_m u_{0m}) \in p^v(x_0)\right\} \quad \text{…. (8.4)}$$

Where $p^v(x_0)$ is output level set consistent with variable returns to scale.

As a linear programming problem $R_0^v(x_0, u_0)$ can be viewed as

$$mR_0^v(x_0, u_0) = Max \sum_{i=1}^{m} \theta_i$$

Such that
$$\left.\begin{aligned} &\sum_{j=1}^{n} \delta_j x_j \leq x_0 \\ &\sum_{j=1}^{n} \delta_j u_j \geq u_0^1 \end{aligned}\right\} \quad \text{…. (8.5)}$$

$$\sum_{j=1}^{n} \delta_j = 1$$

$$\delta_j \geq 0,\ j = 1,2,......n$$

Where $u_0^1 = \left(\theta_1 u_{01,} \theta_2 u_{02,} ....\theta_m u_{0m,}\right)^T$

Russel output technical efficiency can be obtained solving the following optimization problem:

$$R_O^k\left(x_0, u_0\right) = Max\left\{\frac{\sum_{i=1}^{m}\theta_i}{m} : \left(\theta_1 u_{01}, ...\theta_m u_{0m}\right) \in L^k\left(u_0\right)\right\} \quad .... (8.6)$$

Where $L^k\left(u_0\right)$ is consistent with constant returns to scale

As a linear programming problem $R_0^k\left(x_0, u_0\right)$ can be viewed as,

$$mR_0^k\left(x_0, u_0\right) = Max\sum_{i=1}^{m}\theta_i$$

$$\left.\begin{array}{l} \text{Subject to} \sum_{j=1}^{n} \delta_j x_j \leq x_0 \\ \sum_{j=1}^{n} \delta_j u_j \geq u_0^1 \\ \delta_j \geq 0,\ j = 1,2,......n \\ u_0^1 = \left(\theta_1 u_{01,} ....\theta_m u_{0m,}\right)^T \end{array}\right\} \quad .... (8.7)$$

9. Conclusions

Theoretical consideration of technical efficiency has existed since Koopmans defined it for production possibilities for which it is not possible to increase any output without simultaneously increasing any input, ceteris paribus. The nonparametric approach to efficiency measurement known as Data Envelopment Analysis is based on the index of Farrell, which measures radial reduction in all inputs consistent with observed output. Even after Farrell efficiency is achieved, however, there may exist additional slack in individual inputs, suggesting that the Farrell index does not necessarily measure Koopmans inefficiency. To solve this problem, the non-radial Russell measure was introduced. In this paper we proposed various concepts of Russell measure of technical efficiencies in data envelopment analysis,

10. References

1. Afriat S. Efficiency Estimation of Production Functions, International Economic Review. 1972; 13:568-98.
2. Aigner DJ, Chu SF. On Estimating the Industry Production Function’, American Economic Review. 1968; 58:826-39.
3. Aigner DJ, Lovell CAK, Schmidt P. Formulation and Estimation of Stochastic Frontier Production Function Models, Journal of Econometrics. 1977; 6:21-37.
4. Banker Cooper, Seiford Thrall, Chu. Returns to scale in different DEA models, European Journal of Operations Research. 2004; 154:345-362.
5. Charnes A, Cooper WW, Rhodes E. Evaluating program and Managerial Efficiency: An Application of Data Envelopment Analysis to Program Follow Through’, Management Science. 1981; 27:668-697.
6. Coelli TJ. A Computer Program for Frontier Production Function Estimation-Frontier, Version 2.0’, Economics letters. 1992; 39:29-32.
7. Coelli TJ, Battese GE. A Frontier Production Function for Panel Data: With Applications to the Australian Dairy Industry’, Paper presented at the Australian Econometric Conference, Melbourne, 1986.
8. Greene. On the estimation of a flexible frontier production model, Journal of Econometrics. 1980; 13:101-115.
9. Jondrow J, Materov IS, Schmidt P. Lovell CAK. On the Estimation of Technical Inefficiency in the Stochastic Frontier production Function Model, Journal of Econometrics. 1982; 19:233-238.

10. Kumbhakar SC. Production Frontiers, Panel Data and Time Varying Technical inefficiency, Journal of Econometrics. 1990; 46:201-11.
11. Lee LF, Tyler WG. The Stochastic frontier production function and average efficiency: An empirical analysis, Journal of Econometrics. 1978; 17:385-389.
12. Mark Ivaldi, Sylvette Monier-Dilhan, Michel Simioni. Stochastic production frontiers and panel data- A latent variable frame work, European Journal of Operations Research. 1995; 80:534-547.
13. Schmidt P. An error structure for systems of translog cost and share equations, Econometrics workshop paper 8309 (Department of Economics, Michigan State University, Michigan), 1984.
14. Shephard RW. The Theory of Cost and Production Function, Princeton University Press, Princeton, 1970.