Control charts and Cusums under linear trend with Rayleigh distribution

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Abstract
This paper intends to assess the performance of Control Charts and CUSUM charts under linear trend with Rayleigh distribution. A distinct approach, in which the operation of the scheme is regarded as forming a Markovian Chain, is set out. The Run Length properties of Control Charts and CUSUM schemes under conditions of slippage in the mean level by step-change to a new sustained level are well documented. Such control procedures are often used where a genuine out of control signal may result from gradual, rather than step changes. This paper presents, the results of evaluation of run length under linear trend with Rayleigh distribution.

Keywords: Control chart, cumulative sum technique, average run length, run length distribution, linear trend, non-homogeneous Markov chain, transition matrix

1. Introduction
Cumulative Sum Schemes Control Charts were introduced in 1954 by Page (1954). These charts may be used in several situations where a production process is expected to change at an unknown time from an “in control state” to an “out of control state”. As soon as one has evidence that the out of control state has observed one would wishes to stop the production process to take remedial measures. CUSUM schemes have proven to be optimal stopping rules in the sense that to minimize expected run length under the out of control state given that the stopping rules has a fixed expected run length under the in-control state Moustakides (1986) [10].

CUSUM control charts have found an interesting variety of applications since their introduction. Several researchers namely Johnson (1966) [7], Hinkley (1970) [9], Brook and Evans (1972) [11], Ashish and Srivastava (1975) [14], Hawkins (1977) [6], Khan (1978) [8], Koning and Does (1988) [9], Rendel (1990) [12], Rogerson (2006) [13], Cox (2009) [2] have attempted performance of CUSUM charts under various conditions. In most of the research problems the CUSUM chart performance is mainly assessed based on the Average Run Length or its distribution. In other words, the effectiveness of monitoring procedures like Shewart charts with Action limit only, control charts with Warning lines and CUSUM procedures can be demonstrated when there is a slippage in mean level from a target value. This can be done with the help of ARL or other features of run length distributions. The ARL is usually measured on the assumption of step change i.e. abrupt change from the process average. The main purpose of this paper is to assess the performance of control charts and CUSUM charts under linear trend with non-normal distribution namely, Rayleigh. This distribution and its importance is documented. Such control procedures are often used where a genuine out of control signal may result from gradual, rather than step changes. This paper presents, the results of evaluation of run length under linear trend with Rayleigh distribution.

In the subsequent sections we discuss Shewart chart with action line, control chart with warning line and CUSUM procedures.

2. Shewart Control Chart With Action Lines
In the construction of control charts we are using two sets of limits such as action limits or outer limits and warning limits or inner limits. When action lines point plots outside of this limit, a search for an assignable cause is made and corrective action is taken if necessary. Shewart control chart with only action lines, it is denoted by ‘Scheme A’ and specified distributional assumptions, the evaluation of run length properties follows Geometric distribution with parameter $P_A$ that is
ARL = \frac{1}{P_A} \tag{2.1}

Where $P_A$ is the Probability of action limit for a specified process mean.

3. Shewart Control Chart With Warning Lines

In a Shewart control chart with action and warning lines we take decisions with monitoring procedures depend on preceding observations as well as the most recent value. It is denoted by ‘Scheme W’. If one or more points fall between the warning line and the central line or very close to the warning line, we should be suspicious that the process may not be operating properly. One possible action to take when this occurs is to increase the sampling frequency. The use of warning limit can increase the sensitivity of the control chart. The complete run length distribution is obtained by using successive powers of the transition matrix. In particular, the ARL is found to be

\[
\frac{1 + P_W - P_A}{P_A + P_W (P_W - P_A)} \tag{3.1}
\]

Where $P_W$ is the Probability of a violation of warning line which includes more extreme action line violation $P_A$. The action line scheme is having only two states one Transient and another one is absorbing.

4. Transition Matrices for Control Chart and Cusums with Rayleigh Distribution

4.1 The Importance of Rayleigh Distribution

Parametric distributions are often used to model life time and time-to-failure responses. In Probability theory and Statistics, the Rayleigh distribution is a continuous probability distribution. The Rayleigh probability density function is given by

\[
f(x; \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad x \geq 0,
\]

For parameter $\sigma > 0$,

Probability density function

And cumulative distribution function

\[
F(x) = 1 - e^{-x^2/2\sigma^2} \quad \text{For} \quad x \in [0, \infty].
\]

Cumulative distribution function
The Mean of Rayleigh distribution is $\sigma \sqrt{\frac{\pi}{2}}$

The Mode of Rayleigh distribution is $\sigma$

The Variance of Rayleigh distribution is $\frac{4 - \pi}{2} \sigma^2$

Properties

The raw moments are given by: $\mu_k = \sigma^k 2^{k/2} \Gamma(1 + k/2)$ Where $\Gamma(2)$ is the Gamma function. The mean and variance of a Rayleigh random variable may be expressed as:

$\mu(X) = \sigma \sqrt{\frac{\pi}{2}} \approx 1.253 \sigma$, And $\text{var}(X) = \frac{4 - \pi}{2} \sigma^2 \approx 0.429 \sigma^2$.

The mode is $\sigma$ and the maximum pdf is

$f_{\text{max}} = f(\sigma; \sigma) = \frac{1}{\sigma} \exp\left(-\frac{1}{2}\right) \approx \frac{0.606}{\sigma}$

Related distributions

If $R = \sqrt{X^2 + Y^2}$, where $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ are independent normal random variables.

If $R$ is a Rayleigh then $R^2$ has a chi-square distribution with two degrees of freedom: $R^2 \sim \chi^2_2$

If $X$ has an exponential distribution $X \sim \text{Exponential}(\lambda)$, then $Y = \sqrt{2X \sigma^2 / \lambda} \sim \text{Rayleigh} (\sigma)$.

If $R \sim \text{Rayleigh} (\sigma)$, then $\sum_{i=1}^{N} R_i^2$ has a Gamma Distribution with Parameters $N$ and $2\sigma^2$: $\left[ Y = \sum_{i=1}^{N} R_i^2 \right] \sim \Gamma(N, 2\sigma^2)$

Parametric distributions are often used to model life time and time-to-failure responses. In statistical terminology, the Rayleigh distribution is a continuous distribution. For example, the Rayleigh distribution arises when the wind speed is studied with the acceleration of speed towards the orthogonal of two dimensional components. For all these cases we assume that each component is uncorrelated and normally distributed for equal variances. However, the entire phenomena i.e. speed is characterized by a Rayleigh distribution.
4.2 Transition Matrix for Control Chart

The Transition matrices representation for control charts are given below. In case of row labels refer to states at sample (i-1) and column heading to states at sample i. The upper left column partition is the reduced transition matrix after deleting row and column for the absorbing states.

<table>
<thead>
<tr>
<th>Table 4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Transition matrix for “Action only” (Shewart) chart</td>
</tr>
<tr>
<td>clear</td>
</tr>
<tr>
<td>clear</td>
</tr>
<tr>
<td>signal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>W. Transition matrix for “Action and Warning” control chart</td>
</tr>
<tr>
<td>clear</td>
</tr>
<tr>
<td>clear</td>
</tr>
<tr>
<td>Warning</td>
</tr>
<tr>
<td>Signal</td>
</tr>
</tbody>
</table>

4.3 Transition Matrix for Cusum Chart with Rayleigh Distribution

Brooks and Evans (1972) \(^1\) show that CUSUM procedures may be viewed as Markov chains. However, for continuous distributions, it is necessary to consider the discretization for the markov chain representation and the various states then corresponds to values of the CUSUM at any step. For an instance consider a scheme with decision interval H and reference value K are designed to detect upward shift from a target value. A set of \((m + 1)\) states can be interpreted as the CUSUM values of \(H/H, \ldots, H/(m-1)H, \ldots, H/m\), etc.

\[
\begin{align*}
&\leq 0, 0, \frac{H}{m}, \frac{H}{m}, \frac{2H}{m}, \ldots, \frac{(m-1)H}{m} \to \frac{H}{m}, \ldots, 0, 0, \geq H
\end{align*}
\]

(4.3.1)

The last of these states that is violation of the decision interval can be thought of as an absorbing barrier. In the usual Markov chain notation with Transition matrix \(P\) and reduced matrix is one which is obtained from the deletion of the row and column representing the absorbing barrier. The well-known result for obtaining ARL from an initial zero CUSUM is the sum of the elements in the first row of \((I - R)^{-1}\). While considering the states the degree of discretization has some effect on the accuracy of ARL determination. In the present study 20 states transition matrices were used. Thus for \(H=5\) and \(K=0.5\) with \(\mu\) at the target value. The transition matrix in general and for the particular study is shown in tables 4.3 and 5.1 respectively.

<table>
<thead>
<tr>
<th>Table 4.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>C. Transition matrix for CUSUM scheme H, K (m + 1 states)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>(\leq 0)</td>
</tr>
<tr>
<td>CUSUM at</td>
</tr>
<tr>
<td>((i-1))th sample</td>
</tr>
<tr>
<td>(\ldots)</td>
</tr>
<tr>
<td>((m-1))H</td>
</tr>
<tr>
<td>(\geq H)</td>
</tr>
</tbody>
</table>

The entries in the above matrix need some explanation. In the first row, all entries corresponding moves from an initial zero CUSUM, and in the first entry, it indicates that a sample i, the CUSUM remains at or below zero. This means the sample value should not exceed the reference value \(K\). Thus

\[
P_{0 \geq 0} = P(x \leq K) \quad (4.3.2)
\]

For a move from state zero to \(H/m\), the \(i^{th}\) sample must have a value between \(K\) and \((K + H/m)\). So that the subtraction of reference value gives a CUSUM contribution of \(H/m\). After the discretization,

\[
P_{0 \geq 1} = P(x = K + 2H/m) \quad (4.3.3)
\]
5. Run Length Calculation under Linear Trend with Rayleigh Distribution

The method explained for ARL calculations is applicable only under the assumption that the phenomenon or process average undergoes stable distribution. However, in case of linear trend both the distribution functions and transition matrix changes over the time. These changes can be quantifiable for any specific rate of slippage. Here we get a non-homogenous Markov chain and an alternative method of obtaining ARL is essential. This can be easily derived from the procedures for generalising run length distribution under stable conditions.

The entry \( P_{0,m} \) in the transition matrix stands for the probability of occurrence of a signal at the first sample instant. That means \( P_{0,m}^{i} \) stands for element in column 0, row m of \( P^{i} \). This gives probability of signal at the \( i^{th} \) sample. Successive difference between say \( P_{0,m}^{i} - P_{0,m}^{i-1} \) gives the probability of signal at the \( i^{th} \) sample. For an increasing i, we get probability distribution of run length.

For an illustration, consider W chart with action line at 3.09\( \sigma_e \) from a target value and warning limit at 1.96\( \sigma_e \). Let the process changes to a value 1.00\( \sigma_e \) from the target value, then for Rayleigh variable

\[
P_{A} = 1 - \Phi(3.09 - 1) = 0.015299
\]

\[
P_{W} = 1 - \Phi(1.96 - 1) = 0.0146607
\]

The transition matrix in this case is given by

\[
P = \begin{bmatrix}
0.853393 & 0.131308 & 0.015299 \\
0.853393 & 0 & 0.146607 \\
0 & 0 & 1
\end{bmatrix}
\]

The ARL is

\[
\frac{1 + P_{W} - P_{A}}{P_{A} + P_{W} (P_{W} - P_{A})} = 32.7449
\]

The probability of a signal at the first sample \( P_{0,3} \) is 0.015299. Squaring P, we get

\[
\begin{bmatrix}
0.840337 & 0.112058 & 0.047605 \\
0.72828 & 0.112058 & 0.159663 \\
0 & 0 & 1
\end{bmatrix}
\]

The element \( P_{0,3}^{2} \) is 0.0476054. Obviously the probability of a signal a sample 2 is 0.032304. Similarly \( P_{0,3}^{2} \) is 0.076889, gives 0.0061690i the probability run length for three samples.

In the case of non-homogenous transition matrix, it is necessary to multiply original P by new transition matrix obtained after allowing a step change in the mean level. We denote the rate of change by \( \Delta \) and we use \( 1P_0 \), \( 2P_0 \) etc, for the first, second etc samples transition matrices are deduced. In general the Cumulative probability of signal at or before the \( i^{th} \) sample is (0, m)\( \text{th} \) element of the product.

i.e. \( 1P \), \( 2P \), \( 3P \)........ \( iP \)

Individual terms of the run length probability distribution are obtained by successive differences of cumulative probabilities.

Reconsider W scheme with A = 3.09, W = 1.96, with 0 shift, we get

\[
P_{A} = 0.000207
\]

\[
P_{W} = 0.019841
\]

We get

\[
\begin{bmatrix}
0.980159 & 0.017771 & 0.00207 \\
0.980159 & 0 & 0.019841 \\
0 & 0 & 1
\end{bmatrix}
\]

At the second sample, the 0.5\( \sigma_e \) shift i.e. \( \Delta = 0.5 \) gives

\[
P_{A} = 0.005628
\]

\[
P_{W} = 0.053934
\]

We get
Also

\[
\begin{bmatrix}
0.946066 & 0.048306 & 0.005628 \\
0.946066 & 0 & 0.053934 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.944108 & 0.047347 & 0.008545 \\
0.927295 & 0 & 0.025357 \\
0 & 0 & 1
\end{bmatrix}
\]

From the above calculations we get the cumulative probabilities of a signal at sample 2 is 0.008545. Subtracting the (0, m) element of P gives 0.005475. The above computations are presented for the sake of illustration allowing slippages at different levels similar type of calculated and re-designated as \( P \rightarrow P \rightarrow P \rightarrow \ldots \)

For different slippages in mean level, say \( \Delta = 0.01, 0.02, 0.05, 0.1, 0.2 \) are computed and summarized below.

In case of CUSUM scheme the transition matrix can be obtained by making use of the formulae given in section 4.3.3 these are summarised in the following matrix.

#### Table 5.1

<table>
<thead>
<tr>
<th>SHIFT</th>
<th>( P_A )</th>
<th>( P_W )</th>
<th>Signal</th>
<th>( 0.01120045 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000200447</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00021259</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
<tr>
<td>0.00025357</td>
<td>0.02</td>
<td>0.02</td>
<td>0.000200447</td>
<td>0.0946066</td>
</tr>
</tbody>
</table>

6. Run Length Properties of Standard Control Charts under Linear Trend with Rayleigh Distribution

The following table 6.1 gives the average run length and other properties of the run length distributions for three basic control charts namely A scheme, W scheme and CUSUM scheme. In these tables \( \Delta \) values ranges from 0 to 1. Here we note that the shift will be detected rapidly when the trend is greater than 1 standard error per sample in all chart methods.
The entries in the third and fourth column are obtained by using equation (1) and (2) the first column ARL are obtained by using from initial zero CUSUM which is the sum of the elements in the first row of \((I - R)^{-1}\) of respective slippage values. Here CUSUM scheme is operated with \(H = 5, K = 0.5\) under Rayleigh distribution. The following table gives values of ARL and \(\Delta^*\)ARL (displacement ARL) for these basic control charts and the corresponding

\[
\begin{array}{cccc}
\Delta & ARL & \Delta^*ARL & W \\
0 & 504.492 & 431.702 & 400.0484 \\
DEL*ARL & 0 & 0 & \\
0.01 & 494.928 & 422.333 & 390.669 \\
DEL*ARL & 0.00924 & 0 & 0.00924 \\
0.02 & 485.553 & 413.159 & 381.4817 \\
DEL*ARL & 9.71107 & 8.26718 & 7.62963 \\
0.05 & 458.529 & 386.6888 & 355.0345 \\
DEL*ARL & 22.92646 & 19.33444 & 17.75713 \\
0.1 & 416.9404 & 346.1159 & 314.4616 \\
DEL*ARL & 41.69404 & 34.61159 & 31.44616 \\
0.2 & 345.2592 & 276.6742 & 245.02 \\
DEL*ARL & 69.05184 & 55.33484 & 49.00399 \\
0.5 & 199.1828 & 138.9849 & 107.3307 \\
DEL*ARL & 99.59141 & 69.49247 & 53.66535 \\
1 & 86.86585 & 44.40481 & 12.75057 \\
DEL*ARL & 41.36585 & 22.74481 & 12.75057 \\
\end{array}
\]

### Table 6.2: Further Run Length Data for Control Schemes

<table>
<thead>
<tr>
<th>Control Charts</th>
<th>ARL</th>
<th>(\Delta^*)ARL</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = 0.09</td>
<td>304.492</td>
<td>22.92646</td>
</tr>
<tr>
<td>A = 0.39, W = 1.96</td>
<td>367.5596</td>
<td>431.7027</td>
</tr>
<tr>
<td>A = 0.5</td>
<td>345.8578</td>
<td>138.9849</td>
</tr>
<tr>
<td>A = 1</td>
<td>86.86585</td>
<td>44.40481</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cusum Schemes</th>
<th>ARL</th>
<th>(\Delta^*)ARL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H = 0.5)</td>
<td>304.492</td>
<td>22.92646</td>
</tr>
<tr>
<td>(H = 0.5, K = 0.5)</td>
<td>367.5596</td>
<td>138.9849</td>
</tr>
<tr>
<td>(H = 2.5, K = 1)</td>
<td>86.86585</td>
<td>44.40481</td>
</tr>
</tbody>
</table>

7. Conclusions

The various control schemes considered here are, in effect, continuous hypothesis tests. These hypotheses can be stated as

\[H_0: \mu = T\]

Against the alternatives

\[H_1: \mu < T\]

\[H_1: \mu > T\]

"Single-sided" schemes

~17~
$H_0: \mu = T$ "Two-sided" schemes

In real world, it is frequently unknown whether the process averages $\mu$ will change suddenly or gradually. Most of the ARL calculations are based on the one standard deviation schemes under a trend alternative. If trend is expected in the process average, this prior knowledge will be incorporated into the design of control procedures while considering sampling intervals.

In the case of Rayleigh distribution suggest that there exists less difference in performance among schemes A, W and C under the linear trend than under step change conditions. Where as in scheme C with Rayleigh distribution the lower ARL for slippage of 0.2 to 0.1 standard errors is noted. It can be observed that the standard C scheme with Rayleigh distribution gives somewhat quicker response over the range $0.015 \leq \Delta \leq 0.6$ as compared with schemes of A and W schemes over the range $0.03 \leq \Delta \leq 0.3$. These results are broadly compatible for those relating to step changes, in that, for example with ARL $\cong 6$, at $\Delta = 0.3$ for W and C schemes, the process mean must have shifted above two standard errors by the time the trend is detected. Similarly, for A and C shift is about 3 standard error with ARL $\cong 4.9$ at $\Delta = 0.6$. For step changes greater than 2.5, it is observed that lower ARL for W when compared with C scheme. The same situation is prevailed in the case of slippage greater than 2.5. In table 6.2, the values of $\Delta$ ARL gives for further clarification on the point of selection A, W and C alternatives.

8. References