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Mathematical model for minimizing divorce through counseling

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Abstract
In this paper, we propose a mathematical model to examine the impact of counseling in divorce cases. A divorce related hardship/illness forces the individuals to go for counseling. Counseling sections has always help decrease divorce cases and to increase the maximum number of marriages from divorce in our communities. Basic Reproduction Number, stability analysis and its numerical simulation for the proposed model was carried out which proves that counseling sections for the divorced is helpful for improving individuals' health and consequently the community.

Keywords: Divorce related illness, counseling, threshold, marriage, mathematical model

1. Introduction

Divorce according to the oxford dictionaries, is the legal dissolution of a marriage by a court or other competent body. Marriage, on the other hand is a socially or ritually recognized union between spouses that establishes rights and obligations between those spouses, as well as between them and any resulting biological or adopted children and affinity. Haviland *et al.* (2011) [1]

In Ghana, divorce is frowned upon and seen as an element of culture diffusion; no ethnic group has been found to be in favor of divorce. Despite this, divorce cases in Ghana are very high. According to the Regional News of Thursday, 10 July 2014 on 'Ghana Web', six hundred and eighteen (618) customary marriages were dissolved as against a thousand five hundred and eleven marriages registered. The causes of these marriage dissolutions has been enumerated as infertility, financial problems, physical or emotional abuse, infidelity, and even alcoholism.

The negative effects of divorce is enormous and a disadvantage to both parents and children. If parents consider all the negative effects of divorce, I believe the rate of divorce will reduce drastically.

This paper is an improvement on 'Mathematical Model of Divorce Epidemic in Ghana' proposed by Gambrah and Adzadu (2018) [3] and "Divorce Transmission Model" by Gambrah, Abdul-Samad and Adzadu (2018) [6].

In this paper, we will analyze how counseling can decrease the rates of divorce similar to *SEIR* model. The mathematical model, notations along with its parametric values, basic reproduction number are formulated and discussed respectively in Section 2. Stability, where local and global stability are evaluated is in Section 3. Numerical analysis of the results is illustrated in Section 4. Section 5 is the Conclusion.

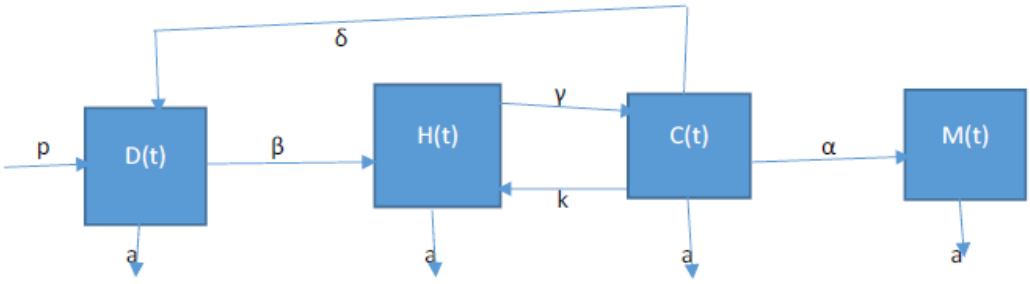
2. Mathematical Model

Here, we formulate a mathematical model for minimizing divorce through counseling using *SEIR* model. Below are the notations along with its description;

D (t): Number of people who are divorced at some time t
H (t): Number of people who are suffering from divorce related hardship/illness at some time t
C (t): Number of people going for counseling at some time *t*
M (t): Number of people who come back to marry at some time *t*

P: New Recruitment Rate
B: Rate of people suffering due to divorce
γ: Rate of people going for counseling due to divorce related hardship
κ: Rate of people who get back to their hardship after counseling
δ: Rate of people who get back to their divorce state after counseling
α: Rate of people who get back to marriage after counseling
a: Natural death rate
Let *N (t)* denotes the sample size of the total population at some time *t*. Here, *N (t)* is divided into four compartments *D (t)*, H *(t)*, *C (t)* and *M (t)* which are described above. Thus, *N (t) = D (t) + H (t) + C (t) + M (t)*.

Below is the schematic diagram to minimize divorce through counseling.



People who are divorced (*D)* get into hardships like how to take care of the children, heartbreaks, emotional and physical distress and the like (*H)* at some stage of life which is defined by the rate β. Rate γ represents the individuals who go for counseling to get rid of this hardships. As, counseling does not ensure 100% remedial of this hardship, the rate k go back to the hardship stage. δ is the rate who again go back to the divorce stage after counseling and α is the rate who quits divorce after counseling. Here, new recruitment rate p and mortality rate a are assumed to be equal.

From the above, a set of non-linear differential equations for minimizing divorce through counseling has been constructed as;

$$\begin{aligned}\frac{dD}{dt} &= p-\beta DH+\gamma C-aD\\ \frac{dH}{dt} &= \beta DH+kC-(\gamma+a)H\\ \frac{dC}{dt} &= \gamma H(1-\alpha)-(a+k)C-\delta C\\ \frac{dM}{dt} &= \alpha\gamma H-aM\end{aligned} \tag{1}$$

With

$$N=D+H+C+M, D>0, H\geq 0, C\geq 0, M\geq 0$$

In the system of equation (1), *N (t)* is constant so we assume that $D(t)+H(t)+C(t)+M(t)=1$.
Also, as the *M* variable does not appear in any of the first three equations from the set of equations (1) we consider the following subsystem of equations;

$$\begin{aligned}\frac{dD}{dt} &= p-\beta DH+\gamma C-aD\\ \frac{dH}{dt} &= \beta DH+kC-(\gamma+a)H\\ \frac{dC}{dt} &= \gamma H(1-\alpha)-(a+k)C-\delta C\end{aligned} \tag{2}$$

Adding the above set of equations (2) we get

$$\frac{d(D+H+C)}{dt}=p-a(D+H+C)-\alpha H\geq 0$$

This gives $\lim_{t\to\infty}\sup\left(D+H+C\right)\le\frac{p}{a}=1$

So, the feasible region for (2) is

$$Y=\left\{\left(D,H,C\right):D+H+C\le 1,D>0,H\ge 0,C\ge 0\right\}$$

Thus, Divorce free equilibrium of system (2) is $E_0=\left(1,0,0\right)$

Now, we are interested in calculating the basic reproduction number which is to be calculated using next generation matrix method, Dikeman *et al.* (2010) [5], Hethcote (2000) [8], Heffernan *et al*. (2005) [7] and Van den Driessche *et al*. (2002) [9]. The next generation matrix method is defined as FV^{-1} where *F* and *V* are both Jacobian matrices of $\Im$ and *v* evaluated with respect to the people suffering from divorce related hardship *(H)* and the one joining counseling *(C)* at the point E_0.

Let $X=\left(H,C,D\right)$

$$\therefore\frac{dX}{dt}=\Im(X)-v(X)$$

Where $\Im(X)$ denotes the rate of new divorce and $v(X)$ denotes the rate of transfer of divorce

Which is given by

$$\Im(X)=\begin{bmatrix}\beta DH\\0\\0\end{bmatrix}\text{ And}$$

$$v(X)=\begin{bmatrix}\left(\gamma+a\right)H-kC\\\left(k+a\right)C-\gamma\left(1-\alpha\right)H+\delta C\\-p+\beta DH-\gamma C+aD\end{bmatrix}$$

Now, the derivative of $\Im$ and *v* at divorce free equilibrium point E_0 gives matrices *F* and *V* of Order 3×3 defined as

$$F=\left[\frac{\partial\Im_i\left(E_0\right)}{\partial X_j}\right]\text{ And }v=\left[\frac{\partial v_i\left(E_0\right)}{\partial X_j}\right]\text{ for }i,j=1,2,3$$

$$\text{Hence }F=\begin{bmatrix}\beta&0&0\\0&0&0\\0&0&0\end{bmatrix}\text{ and }v=\begin{bmatrix}\gamma+a&-k&0\\-\gamma\left(1-\alpha\right)&k+a+\delta&0\\\beta D&-\delta&a\end{bmatrix}$$

Where *v* is non-singular matrix. Thus, the basic reproduction number R_0 which is the spectral radius of matrix FV^{-1} is given as

$$R_0=\frac{\beta\left(k+a+\delta\right)}{\alpha k\gamma+\delta\gamma+\delta a+ka+\gamma a+a^2}.$$

Equating equations (2) to zero, an endemic equilibrium point defined as divorce present equilibrium point E^* is obtained which is as follows:

Divorce present equilibrium is: $E^*=\left(D^*,H^*,C^*\right)$

Where $D^* = \dfrac{p+\delta C}{a+\beta H}$, $C^* = \dfrac{\gamma(1-\alpha)H}{k+a+\delta}$ and $H^* = \dfrac{p(k+a+\delta)(R_0-1)}{R_0\left[a^2+a(k+a+\delta)+\alpha\gamma(k+\delta)\right]}$

3. Stability Analysis

In this section, the local and global stability at E_0 and E^* using the linearization method and matrix analysis are studied.

3.1 Local Stability

Theorem 3.1.1: (stability of E_0) If $R_0<1$ then the divorce free equilibrium point E_0 of System (2) is locally asymptotically stable and if $R_0>1$then it is unstable.

Proof: At point E_0, the Jacobian matrix of the system (2) is

$$J(E_0)=\begin{bmatrix} -a & -\beta & \delta \\ 0 & \beta-\gamma-a & k \\ 0 & \gamma(1-\alpha) & -(k+a+\delta) \end{bmatrix}$$

The characteristic polynomial for the above matrix is

$$\lambda^3+b_1\lambda^2+b_2\lambda+b_3=0$$

Where

$$b_1=k+3a+\delta-\beta+\gamma>0$$

$$b_2=\alpha k\gamma-\beta\delta-\beta k-2a\beta+\delta\gamma+2\delta a+2\gamma a+3a^2>0$$

$$b_3=a\left[\alpha\gamma k-\beta\delta-\beta k-a\beta+\delta\gamma+\delta a+ka+\gamma a+a^2\right]>0$$

By Routh Hurwitz criteria, Allen (2006), if $R_0<1$then it is locally asymptotically stable at E_0 and if $R_0>1$then it is unstable.

Lemma 3.1.2: (stability of E^*) Let K be a real matrix of order 3×3. If *trace* (K), det (K) and Det ($K^{[1]}$) <0 then all the eigen values of the matrix K have negative real parts.

Proof: On linearizing the set of equations (2) at point $E^*=(D^*,H^*,C^*)$ the Jacobian matrix of the system (2) is obtained as follows:

$$J(E^*)=\begin{bmatrix} -a-\beta H^* & -\beta D^* & \delta \\ \beta H^* & \dfrac{-kC^*}{H^*} & k \\ 0 & \gamma(1-\alpha) & -(\delta+k+a) \end{bmatrix}$$

$$\therefore trace\left(J(E^*)\right)=-\beta H^*-2a-\frac{kC^*}{H^*}-k-\delta<0$$

$$\det\left(J(E^*)\right)=-\left[H^*\left\{D^*\beta^2(\delta+k+a)+\beta\delta\gamma(\alpha-1)+\beta k\gamma(\alpha-1)+C^*\beta k(\delta+k+a)+k\gamma a(\alpha-1)+\frac{akC^*}{H^*}(\delta+a+k)\right\}\right]$$
$$=\beta a(\delta+k+a)(-1+D^*)<0$$

Now, the second additive compound matrix of $J(E^*)$, which is given by $J^{[2]}(E^*)$ is as follows:

$$J^{[2]}\left(E^*\right)=\begin{bmatrix} -\left(a+\beta+\dfrac{kC^*}{H^*}\right) & k & -\delta \\ \gamma\left(1-\alpha\right) & -\left(\delta+k+2a+\beta H^*\right) & -\beta D^* \\ 0 & \beta H^* & -\left(\dfrac{kC^*}{H^*}+\delta+k+a\right) \end{bmatrix}$$

$$\det\left(J^{[2]}\left(E^*\right)\right)=-\left[\beta^2\left(\beta D+\delta+k+a\right)\right]I^{*2}-\left[\beta\left(D\beta a+C\beta k-\alpha\delta\gamma+\delta^2+2\delta k+\delta\gamma+4\delta a+k^2+4ka+3a^2\right)H^*\right]$$
$$-\left[DC\beta^2k+2C\beta k\delta+2C\beta k^2+4C\beta ka+\alpha\delta\gamma k+\alpha k^2\gamma+\alpha k\gamma a+\delta^2a-\delta k\gamma+2\delta ka+3\delta a^2-k^2\gamma+k^2a-k\gamma a+3ka^2+2a^3\right]$$
$$-\frac{1}{H^*}\left[C^2\beta k^2+C\alpha k^2\gamma+C\delta^2k+2C\delta k^2+4C\delta ka+Ck^3-Ck^2\gamma+4Ck^2a+4Cka^2\right]$$
$$-\frac{1}{H^*}\left[C^2\delta k^2+C^2k^3+2C^2k^2a\right]$$
$$<0$$

Hence, E^* is locally asymptotically stable by above lemma.

3.2 Global Stability

Theorem 3.2.1: (stability of E_0) If $\beta \le \alpha\gamma$ then E_0 is globally asymptotically stable.

Proof: Consider the Lyapunov function

$$D = H + C$$

$$\frac{dD}{dt}=-\left(\gamma+a\right)H+\beta DH+kC+\gamma\left(1-\alpha\right)H-\left(k+a\right)C-\delta C$$
$$=-a\left(H+C\right)+H\left(\beta-\gamma\alpha\right)-\delta C\le 0$$

We have $\dfrac{dD}{dt}<0$ for $\beta D\le\gamma\alpha$

But we have noted that D <1so $\dfrac{dD}{dt}<0$ for $\beta\le\alpha\gamma$ and $\dfrac{dD}{dt}=0$ when $T_B=M=0$.

By LaSalle's Invariance Principle [La Salle (1976)] [2], E_0 is globally asymptotically stable.

Theorem 3.2.2: (stability of E^*) Consider a piecewise smooth vector field $g\left(D,H,C\right)=\left\{g_1\left(D,H,C\right),g_2\left(D,H,C\right),g_3\left(D,H,C\right)\right\}$ On Y^* that satisfies the condition $\left(Cur\lg\right).\vec{n}<0$, $g.f=0$ inside Y^*, where $f=\left(f_1,f_2,f_3\right)$ is a Lipschitz continuous field inside Y^*, $\vec{n}$ is a normal vector to Y^* and $Cur\lg=\left(\dfrac{\partial g_3}{\partial H}-\dfrac{\partial g_2}{\partial C}\right)\hat{i}-\left(\dfrac{\partial g_3}{\partial D}-\dfrac{\partial g_1}{\partial C}\right)\hat{j}+\left(\dfrac{\partial g_2}{\partial D}-\dfrac{\partial g_1}{\partial H}\right)\hat{k}$.

Then, the system of differential equations $D=f_1, H=f_2, C=f_3$ *has* no homoclinic loops, periodic solutions and oriented phase polygons inside Y^*.

Proof: Suppose $Y^*=\left\{\left(D,H,C\right):D+\left(\dfrac{a+\alpha\gamma}{a}\right)H+C=1, D>0, H\ge 0, C\ge 0\right\}$. Also, it can be proved that Y^* is a subset of Y, Y^* is positively invariant and endemic equilibrium E^* belongs to Y^*. Let f_1, f_2 and f_3 represents the right-hand side of equations in the set of equations (2) respectively. Using $D+\left(\dfrac{a+\alpha\gamma}{a}\right)H+C=1$ to write f_1, f_2 and f_3 in the equivalent forms, we get

$$f_1\left(D,H\right)=p-\beta DH+\delta C-aD$$

$$f_1(D,C) = p - \beta D\left[(1-C-D)\left(\frac{a+\alpha\gamma}{a}\right)\right] + \delta C - aD$$

$$f_2(D,H) = \beta DH + kC - (\gamma + a)H$$

$$f_2(H,C) = \beta H\left[1 - C - \left(\frac{a+\alpha\gamma}{a}\right)H\right]H + kC - (\gamma + a)H$$

$$f_3(D,C) = \gamma(1-\alpha)\left[(1-C-D)\left(\frac{a+\alpha\gamma}{a}\right)\right] - (a+k)C - \delta C$$

$$f_3(H,C) = \gamma(1-\alpha) - (a+k)C - \delta C$$

Suppose $g = (g_1, g_2, g_3)$ is a vector field such that

$$g_1 = \frac{f_3(D,C)}{DC} - \frac{f_2(D,H)}{DH}$$

$$g_2 = \frac{f_1(D,H)}{DH} - \frac{f_3(H,C)}{HC}$$

$$g_3 = \frac{f_2(H,C)}{HC} - \frac{f_1(D,C)}{DC}$$

The alternate form of f_1, f_2 and f_3 are equivalent

$$g.f = g_1 f_1 + g_2 f_2 + g_3 f_3 = 0$$

Normal vector $\vec{n} = \left(1, \frac{a+\alpha\gamma}{a}, 1\right)$

$$Curl\vec{g} = \left[\frac{-k}{H^2} - \frac{\delta}{DH} - \frac{\gamma(1-\alpha)}{C^2}\right]\hat{i} - \left[\frac{\beta}{C} + \frac{p}{D^2C} + \frac{\delta}{D^2} - \frac{\beta}{C}\left(\frac{a}{a+\alpha\gamma}\right)\right]\hat{j} + \left[-\frac{p}{D^2H} - \frac{\delta C}{D^2H} - \frac{k}{D^2H} + \frac{k}{H^2}\right]\hat{k}$$

$$(Curl\vec{g}).\vec{n} = -\frac{\delta}{DH} - \frac{a+\alpha\gamma}{D^2C} - \frac{\delta(a+\alpha\gamma)}{D^2H} - \frac{\gamma(1-\alpha)}{DC^2} - \frac{p}{D^2H} - \frac{\delta C}{D^2H} - \frac{k}{DH^2} - \frac{\beta\alpha\gamma}{aC} < 0$$

So, the system (2) has no homoclinic loops, periodic solutions and oriented phase polygons in the interior of Y^*.
Therefore E^* is globally asymptotically stable in the interior of Y^*.

4. Numerical Simulation

In this section, numerical results with their interpretation are described which will help us to know the impact of counseling for minimizing divorce. The table below will be used.

Table 1: Notations and its Parametric Values

S/N	Notation	Parametric Values
1.	D(t)	90.00
2.	H(t)	40.00
3.	C(t)	34.00
4.	M(t)	14.00
5.	p	0.21
6.	β	0.21
7.	γ	0.12
8.	κ	0.09
9.	δ	0.10
10.	α	0.15
11.	a	0.021

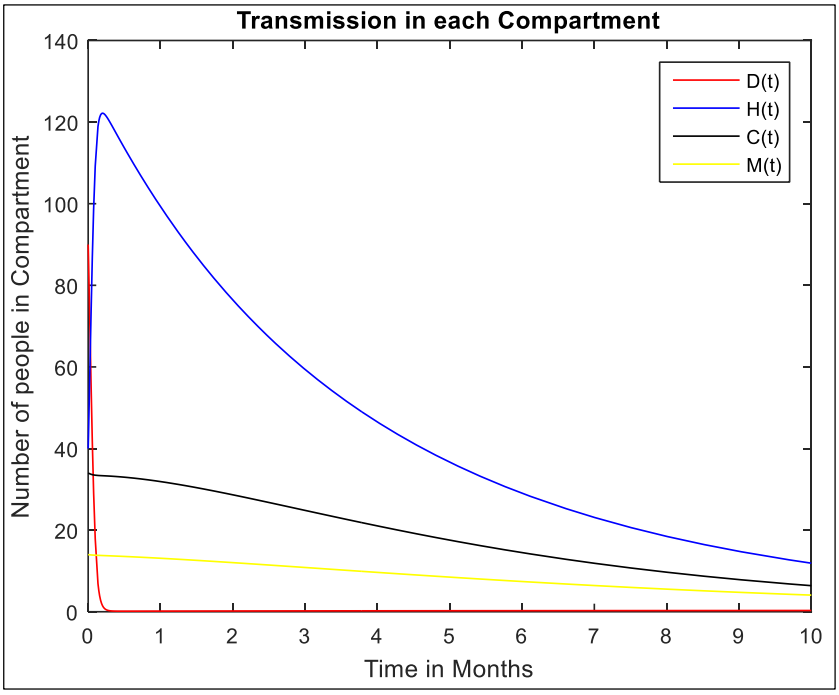


Fig 1: Transmission of people in each compartment

Figure 1 shows that in the initial stage divorce decreases which means that, the hardship due to divorce is increasing simultaneously. This increase in hardship pushes the individual to go for counseling which is increasing in the first few months but this decreases and becomes stable after some months which implies that, it forces maximum number of people to marry, but as counseling decreases remarriage also decreases.

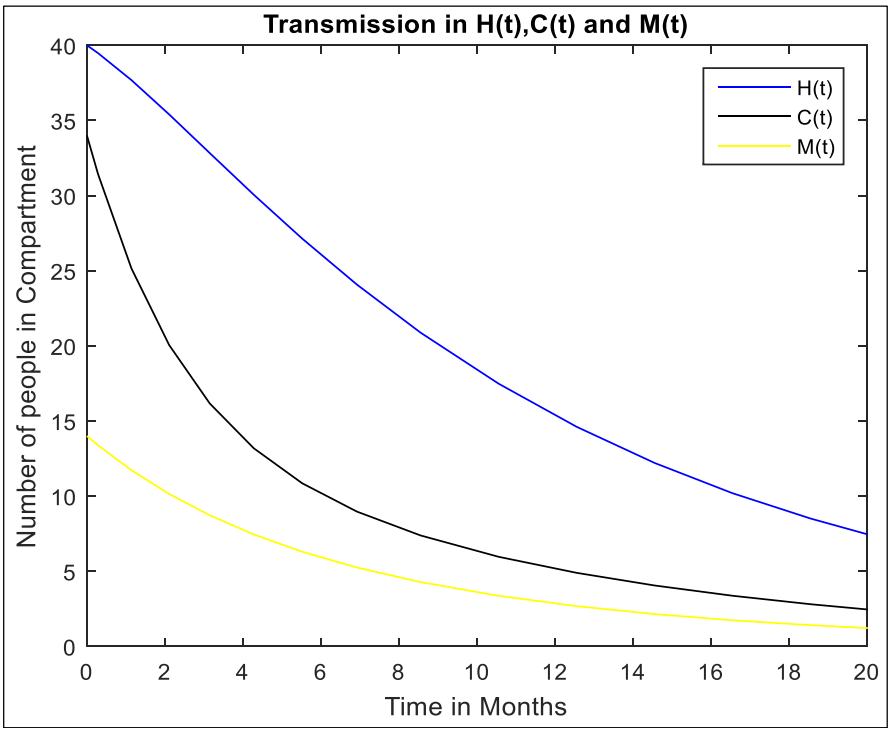


Fig 2: Transmission of H, C and M

Figure 2 shows that hardship caused due to divorce has made them adapt a way of moving from the divorce by going for counseling and this has proved to be successful because as time goes on all three goes to zero.

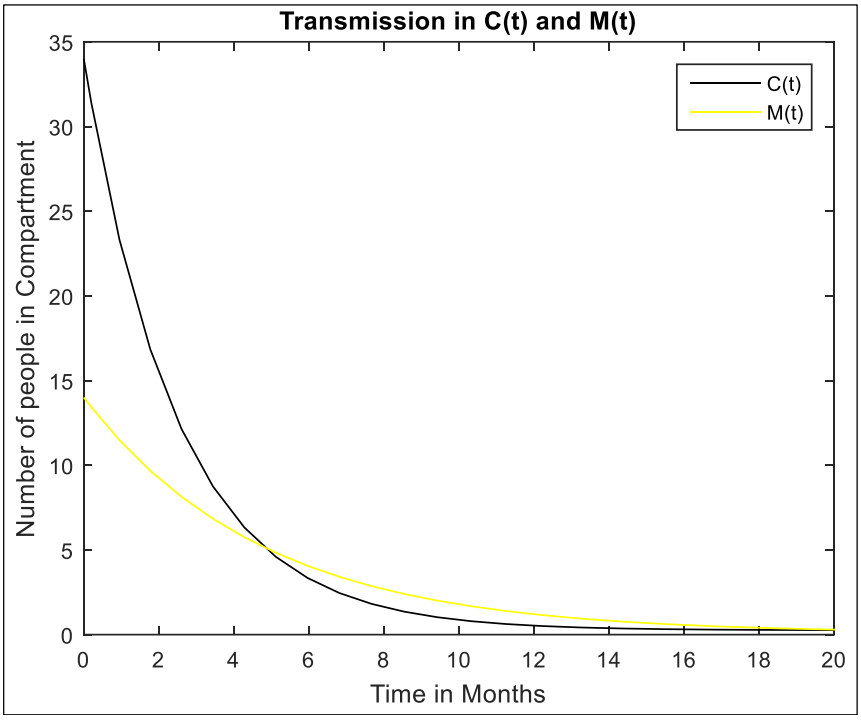


Fig 3: Transmission of C and M

Figure 3 shows that the impact of counseling is truly high to increase the number of people who go into marriage again. Counseling has proven to played a vital role in minimizing divorce in our society

5. Conclusion

A mathematical model has been formulated as a system of nonlinear ordinary differential equations to study the impact of counseling on people who go for divorce. The systems were proven to be locally and globally asymptotically stable at both equilibrium points. A basic reproduction number has been calculated at a divorce free equilibrium point and is equal to 0.34508 which shows that with counseling, divorce will not be and epidemic in the society. Numerical Simulation was carried out to examine the results of the compartments which shows that hardship/stress due to divorce can earlier be minimize through counseling.

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