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# Soliton solution of korteweg-de vries equation

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**Abstract**
The Korteweg-de Vries (K-dV) equation plays an important role in studying of the propagation of low amplitude water waves in shallow water bodies and the remarkable discovery of soliton solution K-dV equation that leads to solitary waves. The importance of soliton solution one can predict how energy is transported from one part of medium to another and soliton carries energy away from its sources. Soliton solution has become a breakthrough in mechanics, nonlinear analysis and many developments in algebra, analysis, geometry and physics. We present the analytic solution of K-dV equation and then using finite element analysis to predict the soliton behavior in shallow water bodies. The propagation of long water wave equation is close to the soliton solution of our said equation has been investigated in this study. The valid analytical solution for k-dv equation is restricted to time and hence close to the initial position and time as well. Finite element analysis that leads to the soliton solution of k-dv equation has been observed and compare among the graphical representation in this research.

**Keywords:** Soliton solution, propagation, shallow water, amplitude

## Introduction

The Korteweg-de Vries (K-dV) equation was not until the 1960s and the advent of modern computers that the significance of Scott Russell's discovery in physics, electronics, biology and especially Fiber optics started to become understood, leading to the modern general theory of solitons. Note that solitons are, by definition, unaltered in shape and speed by a collision with other solitons. So solitary waves on a water surface are near-solitons, but not exactly – after the interaction of two (colliding or overtaking) solitary waves, they have changed a bit in amplitude and an oscillatory residual is left behind. Klaus Brauer (2000, 2014), who has shown how to find an exact solution to the K-dV equation and who has used the tool of Bäcklund transform [7-9] to obtain an analytical solution and who - in addition to that - performs non-linear superposition of two, three and more solutions corresponding to two, three or more soliton waves by using Bäcklund transform again. Numerical study using finite element analysis of K-dV equation will be done in this study.

## Mathematical formulation

The usual expression of K-dV equation is

$$U_t(\mathrm{x,t}) + 6U(\mathrm{x,t})\,U_x(\mathrm{x,t}) + U_{xxx}(\mathrm{x,t}) = 0 \qquad (1)$$

Where $U_t(\mathrm{x,t}) = \dfrac{\partial U(\mathrm{x,t})}{\partial t}$, $U_x(\mathrm{x,t}) = \dfrac{\partial U(\mathrm{x,t})}{\partial x}$, $U_{xx}(\mathrm{x,t}) = \dfrac{\partial^2 U(\mathrm{x,t})}{\partial x^2}$, $U_{xxx}(\mathrm{x,t}) = \dfrac{\partial^3 U(\mathrm{x,t})}{\partial x^3}$

The above equation described by Russell can be expressed by a non-linear Partial Differential Equation of the third order. In practical applications where the PDE describes a dynamic process one of the variables has the meaning of the time denoted by t and the other normally only up to 3 variables have the meaning of the space denoted by $x, y$ *and* $z$. $x$ describes the only one space variable and $U(x,t)$ describes the elongation of the wave at the place $x$ at time $t$.

**Exact solution of K-dV equation**

Travelling waves as solutions to the Korteweg de-Vries equation (K-dV) which is a non-linear Partial Differential Equation (PDE) of third order have been of some interest already since 150 years. We can present an analytical exact result to the K-dV equation by elementary operations.

Let a trial solution

$$U(x,t) = z(x-\beta t) \equiv z(\zeta) \tag{2}$$

Just denoting the parameter $c$ above by $\beta$ and the function $f$ by $z$.

Substituting the trial solution (3) into (1), we get the ODE

$$-\beta\frac{dz}{d\zeta}+6z\frac{dz}{d\zeta}+\frac{d^3z}{d\zeta^3}=0$$

Integrating both sides we've

$$\int(-\beta\frac{dz}{d\zeta}+6z\frac{dz}{d\zeta}+\frac{d^3z}{d\zeta^3})d\zeta=c_1$$

$$\Rightarrow -\beta z+3z^2+\frac{d^2z}{d\zeta^2}=c_1$$

Where $c_1$ is constant of integration.

Multiplication on both sides with $\frac{dz}{d\zeta}$

$$-\beta zz'+3z^2z'+z''z'=c_1z'$$

Where $z'$ and $z''$ denote the first and second derivatives of the function $z$ with respect to $\zeta$.

Integration on both sides with r. to $\zeta$

$$\Rightarrow -\beta\frac{z^2}{2}+z^3+\frac{1}{2}(\frac{dz}{d\zeta})^2=c_1z+c_2$$

Where $c_2$ in the constant of integration

Now it is required that in case $x\to\pm\infty$

We should have $z\to 0, \frac{dz}{d\zeta}\to 0, \frac{d^2z}{d\zeta^2}\to 0$

from these requirements it follow $c_1=c_2=0$

From equation (5) we can be written as

$$-\beta\frac{z^2}{2}+z^3+\frac{1}{2}(\frac{dz}{d\zeta})^2=0$$

By separation of variable with integration on both sides we get.

$$\int_0^z\frac{d\zeta}{\zeta\sqrt{\beta-2\zeta}}=\int_0^\zeta d\zeta \tag{3}$$

The integration of the left hand side of (3) can be done by using transformation

$$\zeta=\frac{1}{2}\beta\, Sech^2\, w \tag{4}$$

Using (2) we finally get

$$U(x,t)=\frac{\beta}{2}\,\mathrm{sech}^2\left[(\frac{\sqrt{\beta}}{2}(x-\beta t)\right] \tag{5}$$

Instead of (4) we select the transformation

$$\zeta=-\frac{1}{2}\beta\, cosch^2\, w \tag{6}$$

In the same way as we obtained the solution (12) we will obtain another solution which is

$$U(x,t)=-\frac{\beta}{2}\,\mathrm{csch}^2(\frac{\sqrt{\beta}}{2}(x-\beta t)) \tag{7}$$

The solution (14) is an irregular solution to the K-dV Equation. It has a singularity for vanishing argument of the cosech-function, i.e. for the line in the $x-t$ plane.

Equations (5) and (7) are solutions to the original K-dV Equation [4, 5].

**Method of calculation**

Many numerical methods are being used to find the approximate solutions which are very accurate for the evolutionary partial differential equations giving to solitary wave solutions [19]. The solution of K-dV equation is given by

$$u_t(\mathrm{x,t})+6u(\mathrm{x,t})\,u_x(\mathrm{x,t})+u_{xxx}(\mathrm{x,t})=0 \quad in \quad \Omega=[-8,8]$$

*b.c.s*

$$u(-8,\mathrm{t})=\mathrm{u}(8,\mathrm{t})$$

$$u(\mathrm{x},0)=\frac{\beta}{2}sceh^2\frac{\beta}{2}x$$

In order to have a real solution the quantity $\beta$ must be a positive number. It is seen from boundary conditions for $\beta>0$ the solitary wave moves to the right. The second point is that the amplitude is proportional to the speed which is indicated by the value of $\beta$. Thus larger amplitude solitary waves move with a higher speed than smaller amplitude waves. In particular, in this study we consider $\beta=4$.

**Result and Discussion**

To obtain the numerical solutions, we used periodic boundary conditions for a region $0\le t\le 1.18$ and $x$ is restricted in the interval [-8, 8] in most of the evolutionary K-dV equation. Then using finite element method we obtain the desired solution. For discussion we have represented the result graphically. The results of the computations, however, show graphical changes in the below mentioned quantities to time $t=0$ have been reached and after this at $t=(0,1.18)$ graphical change shown. Thus the solution for time $t=(0,1.18)$ is essential soliton solutions of the K-dV equation. Here, figure (1-5) show the soliton spilt into two waves one big and one small as the wave is too big for the depth of the water at time t=0 to 0.38. Fig. 1 represents that the soliton is

stable at $t = 0.0$. Then at time$t = 0.02$, the soliton obtains itsmaximum state compared with the depth of the water and it begins to split at time$t = 0.04$. The height of the soliton is increasing and at the same time the length of soliton is decreasing. The splitting situation of the soliton is also shown in fig. (2-4). The soliton is completely splitting into two solitons at time $t = 0.32$. One is big and other is small. The big soliton is obtained its maximum height but its length is minimum. Then two solitons are moving forward at time $t = 0.34,0.36,0.38$. These phenomena are shown in fig. (5). In figure (6-7), one soliton is leaving in the domain $\Omega = [-8,8]$ and another is entering in the domain$\Omega = [-8,8]$. Fig.8 represents one big soliton has entered in the domain$\Omega = [-8,8]$. It is also going close to the small soliton. The soliton is obtained the closest position from fig. (9-11). At the same time the length and height remain same. The big soliton is overtaking the small solitonand the height of the two soliton is same but their length is increasing in fig. (12) & (13). Fig.14 shows that the height is started to decrease and the length also increasing. The big soliton completely overtaken small soliton and obtained its maximum wavelength with minimum height is shown in fig. (15).

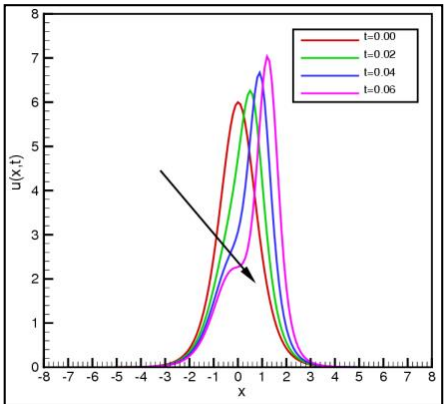


**Fig 1**

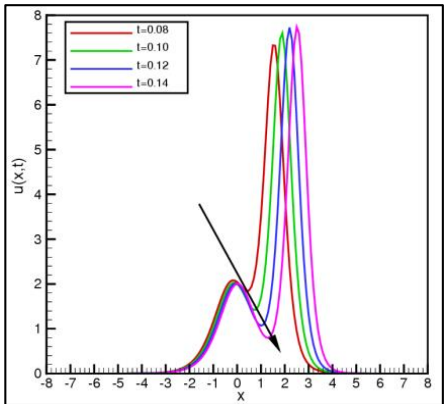


**Fig 2**

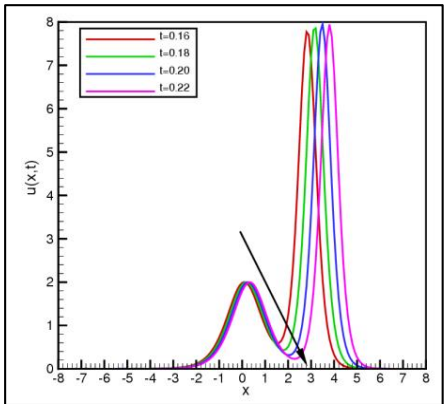


**Fig 3**

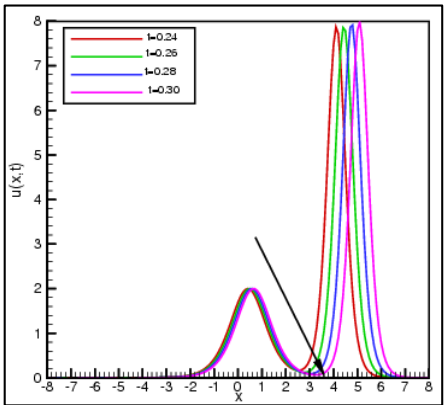


**Fig 4**

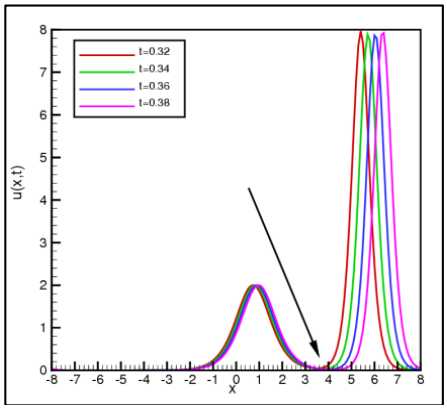


**Fig 5**

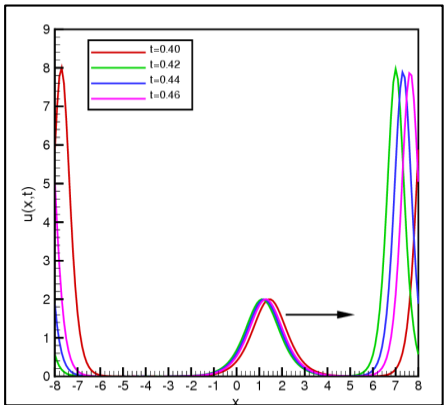


**Fig 6**

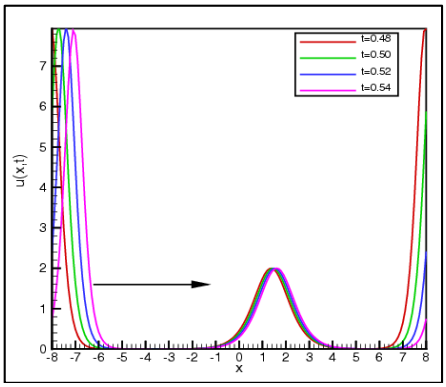


**Fig 7**

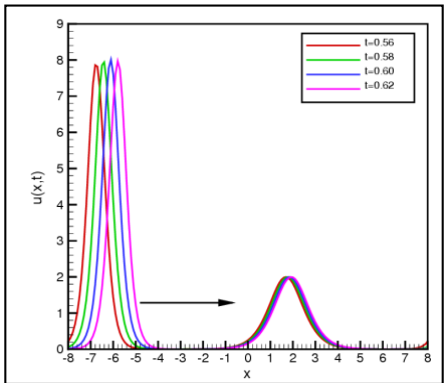


**Fig 8**

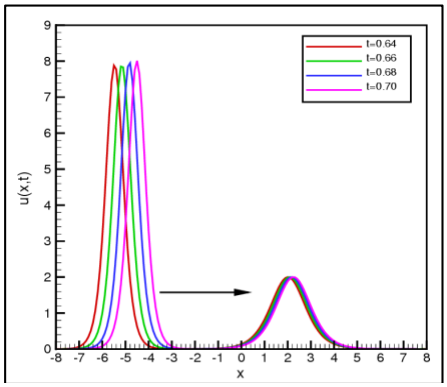


**Fig 9**

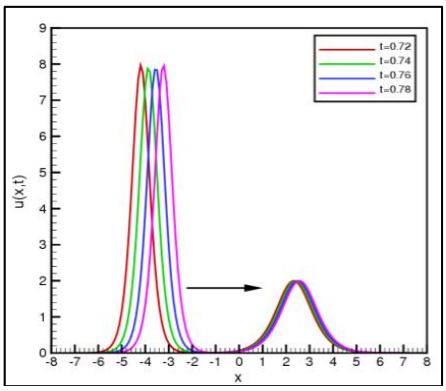


**Fig 10**

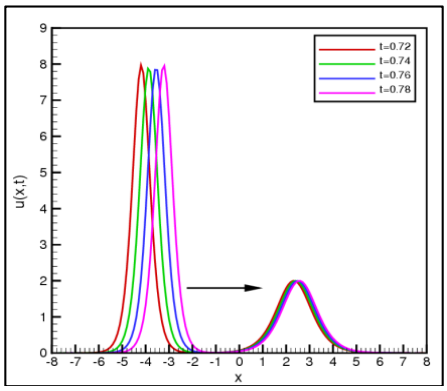


**Fig 11**

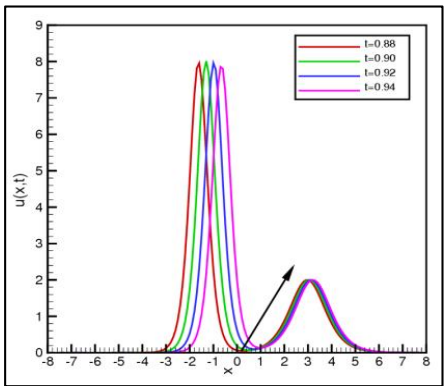


**Fig 12**

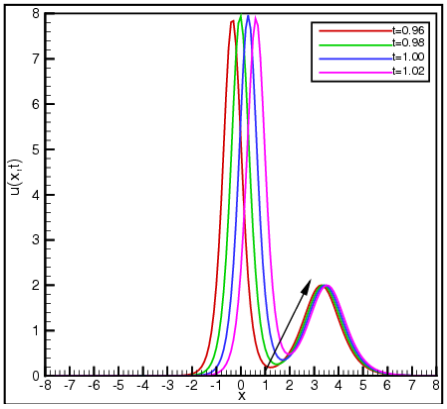


**Fig 13**

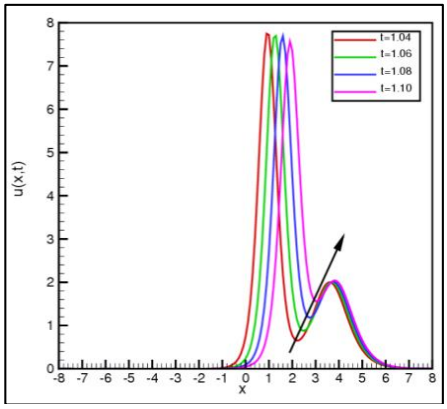


**Fig 14**

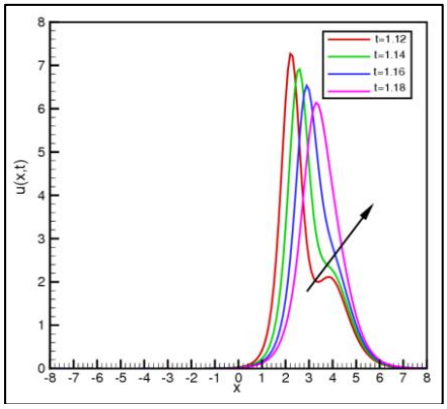


**Fig 15**

## Conclusion

The major finding of our study is given below:

- The waves are stable, and can travel over very large distances (Normal waves would tend to either flatten out, or steepen and topple over)
- The speed depends on the size of the wave, and its width on the depth of water.
- Unlike normal waves they will never merge—so a small wave is overtaken by a large one, rather than the two combining.
- If a wave is too big for the depth of water, it splits into two, one big and one small.

These behaviors are also observed by John Scott Russell. So we can say that our investigations are accurate.

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