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Bernard Mutuku Nzimbi
School of Mathematics, College of Biological and Physical Sciences, University of Nairobi, Nairobi, Kenya

Stephen Wanyonyi Luketero
School of Mathematics, College of Biological and Physical Sciences, University of Nairobi, Nairobi, Kenya

Corresponding Author:
Bernard Mutuku Nzimbi
School of Mathematics, College of Biological and Physical Sciences, University of Nairobi, Nairobi, Kenya

A note on α -Isometries and self-similarity of operators on Hilbert spaces

Bernard Mutuku Nzimbi and Stephen Wanyonyi Luketero

Abstract
In this paper we investigate α -isometries and related classes of operators. We will also introduce and study the notions of similitudes, self-similarity and α -metric equivalence relations of operators. It will be shown that self-similarity implies α -metric equivalence of operators. We will characterize α -isometric and α -unitary and prove that quasisimilar α -isometries are unitarily equivalent.

Keywords: α -isometry, α -unitary, self-similarity, similitude, α -projection, α -metric equivalence +

1. Introduction

Let H denote a Hilbert space and $B(H)$ denote the Banach algebra of bounded linear operators. If $T \in B(H)$, then T^*denotes the adjoint of T while $Ker(T), Ran(T), \overline{M}$and $M^{\perp}$ stands for the kernel of$T$, range of$T$, closure of M and orthogonal complement of a closed subspace M of$H$, respectively. We denote by $\sigma(T), \|T\|, r(T)$ and $W(T)$ the spectrum, norm, spectral radius of T and numerical range of T, respectively. Recall that an operator $T \in B(H)$ is normal if $T^*T = TT^*$, unitary if $T^*T = TT^* = I$, a projection(or idempotent) if $T^2 = T$, an orthogonal projection if $T^2 = T$and $T^* = T$, an isometry if $T^*T = I$ (equivalently$\|Tx - Ty\| = \|x - y\|$, for all $x, y \in H$), a partial isometry if $TT^*T = T$, quasinormal if $(T^*T)T = T(T^*T)$, a scalar if $T = \alpha I$ where $\alpha \in \mathbb{C}$, a contraction if $\|Tx\| \leq \|x\|$, for all$x \in H$.

Two operators $A \in B(H)$ and $B \in B(K)$ are said to be *similar* if there exists an invertible operator $N \in B(H, K)$ such that $NA = BN$or equivalently$A = N^{-1}BN$, and are *unitarily equivalent* if there exists a unitary operator $U \in B_+(H, K)$ (Banach algebra of all invertible operators in$B(H)$) such that $UA = BU$ (i.e.$A = U^*BU$equivalently, $A = U^{-1}BU$). An operator $X \in B(H, K)$ is a quasiaffinity if it is injective with dense range. Two operators $A \in B(H)$ and $B \in B(K)$ are quasisimilar if there exist quasiaffinities $X \in B(H, K)$ and $Y \in B(K, H)$ such that $XA = BX$ and $AY = YB$. Two operators $A, B \in B(H)$ are said to be metrically equivalent if$\|Ax\| = \|Bx\|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$for all$x \in H$, which is equivalent to having $A^*A = B^*B$). The notion of metric equivalence of operators was introduced by Nzimbi *et al.* [6]. Clearly similarity, unitary equivalence and metric equivalence are equivalence relations on$B(H)$.

A *similitude* is an operator $T \in B(H)$ that is a positive scalar multiple of an isometry (equivalently, if $\|Tx - Ty\| = \alpha\|x - y\|$, for all $x, y \in H$ and $\alpha > 0$). Clearly every isometry is a similitude.

Similitudes are obtained as compositions of rotation, translation and scaling (dilation) operators. Such operators are similarities in the geometric sense, in that they map sets to similar sets and so preserve shapes up to a scaling factor. Similitudes are extensions of isometries. The theory of isometries has been extensively studied (see [2], [4] and [5] for more exposition).

2. Main results

Theorem 2.1 The product of similitudes is a similitude.

Proof. Suppose $A, B \in B(H)$ such that $A = \alpha V$ and$B = \beta W$, where $\alpha, \beta > 0$ and $V, W$are isometries. Then $AB = \alpha\beta VW$. The rest of the proof follows from the fact that the product of isometries is an isometry.
Theorem 2.1 says that the set of similitudes is a semigroup.

Theorem 2.2 If $A, B \in B(H)$ are quasisimilar similitude operators, then they are similar.
Proof. Suppose, without loss of generality, that $A = \alpha U$and $B = \alpha V$, where U, V are isometries and $\alpha > 0$. Using the fact that every isometry is quasinormal and ([9], Proposition 4.2), we conclude that any quasisimilar isometries in a Hilbert space are unitarily equivalent. Suppose$U = WVW*$, where $W$is unitary. Then$A = \alpha U = \alpha W(\frac{1}{\alpha}B)W* = WBW* = WBW^{-1}$. This proves the claim.
A consequence of Theorem 2.2 is that quasisimilar similitudes are unitarily equivalent.

Definition 2.3 Two operators $A, B \in B(H)$ are said to be *self-similar* denoted by $A \sim^{s.s} B$ if there is a similitude $S \in B(H)$ such that$B = SA$.
A closed subspace $M \subseteq H$is said to be *self-similar* if it is invariant under a similitude$T$.
Remark. We note that the trivial subspaces $M_1 = \{0\}$and $M_2 = H$are invariant under any bounded linear operator on $H$and hence are self-similar.

Definition 2.4 An operator $T \in B(H)$ is called an ε *−isometry* if $\big|\|Tx - Ty\| - \|x - y\|\big| \leq \varepsilon$, where $\varepsilon \geq 0$and all $x, y \in H$and $T(0) = 0$.
In particular, if$\varepsilon = 0$, then the class of 0 −isometries contains the class of isometries and contraction operators. A similitude need not be invertible, but if it is invertible, then it is a multiple if a unitary operator.

Corollary 2.5 If $A, B \in B(H)$ are invertible self-similar operators, then$B = WA$, where$W = \alpha U$, with $\alpha > 0$.

Definition 2.6 Two operators $A, B \in B(H)$ are said to be α −metrically equivalent if there exists an $\alpha > 0$ such that $A^*A = \alpha^2 B^*B$.
Clearly metrically equivalent operators are α −metrically equivalent.

Theorem 2.7 If$A, B \in B(H)$ areα −metrically equivalent invertible operators then $|B| = \alpha|A|$.
Proof. Using the polar decomposition $B = U|B|$ and $A = V|B|$, where $U, V$are unitary operators, $B^*B = \alpha^2 A^*A$ implies that $|B|^2 = \alpha^2|A|^2$. Taking square roots both sides gives the required equality.

Theorem 2.8 [5] An operator T is a multiple of an isometry if and only if T^*T is a scalar.

Theorem 2.9 If T is a similitude, then $\sigma(|T|) = \{\alpha\}$ for some $\alpha > 0$.
Proof. This follows from the fact that T is a similitude if and only if $|T| = \alpha I$, for some $\alpha > 0$.

Definition 2.10 An operator $T \in B(H)$ is called a α *−projection* if $T = \alpha P$, for some projection $P \in B(H)$.
An operator $T \in B(H)$ is called a α *−isometry* if $T = \alpha V$, where $V$is an isometry and $\alpha \in \mathbb{C}$.
It is clear that if $A \in B(H)$ is an α *−isometry* and $U$is an isometry, then $B = UAU^*$ is a α −projection. It is also clear that an α −projection is a projection if $\alpha \in \{0,1\}$.
It is also clear that a similitude is an α −isometry. In fact the following inclusion holds:
$Similitude \subset \varepsilon - isometry \subset \alpha - isometry.$

Theorem 2.11 If $A \in B(H)$ is unitarily equivalent to a similitude, then it is a similitude.
Proof. Suppose that $A = UBU^*$, where $B = \alpha V$ with V isometric. Then a simple computation shows that $A = UBU^* = U(\alpha V)U^* = \alpha UVU^*$. Now, if we let $W = UVU^*$, it is clear that $W$is an isometry. Therefore
$A = \alpha W$. This establishes the claim.

Example 1 For $\alpha > 0$, the operator Tacting on $\ell^2(\mathbb{N})$ defined by $T(x_1, x_2, x_3, \cdots) = (0, \alpha x_1, \alpha x_2, \alpha x_3, \cdots)$ is a similitude. Clearly, $T = \alpha V$, where $V$is the unilateral shift on $\ell^2(\mathbb{N})$.

Example 2 The operator $S: \mathbb{C}^2 \to \mathbb{C}^2$ defined by $S(re^{i\theta}) = \frac{1}{\sqrt{2}} re^{i(\theta - \frac{\pi}{4})}$is an α −isometry (indeed, a similitude), with $\alpha = \frac{1}{\sqrt{2}}$. This operator is a rotation/translation (isometry) followed by a re-scaling.

Example 3 The operator $S: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(Sf)(x) = \frac{1}{2} f(x - a)$, where $a \in \mathbb{R}$is a similitude. This operator is a re-scaling of a right translation.

Proposition 2.12 If $T \in B(H)$ is an α −isometry, then
a. $T$is injective if and only if $\alpha \neq 0$.
b. $\|T\| = |\alpha|$.

Proposition 2.13 If $T \in B(H)$ is an invertible similitude, then T^{-1} is a similitude.

In fact the set of similitudes is a multiplicative group.

Theorem 2.14 Self-similarity is an equivalence relation in the Banach subalgebra of invertible operators.
Proof. Recall that two operators $A, B \in B(H)$ are self-similar if there exists a similitude $S \in B(H)$ such that $A = SB$.
Clearly, $A = IA$. Thus $A \overset{s.s}{\sim} A$. Suppose $A, B \in B(H)$ are invertible and $A \overset{s.s}{\sim} B$. Then there exists an invertible similitude$S \in B(H)$ such that $A = SB$. Using Proposition 2.13, we have that $B = S^{-1}A$. This shows that $B \overset{s.s}{\sim} A$.
Now, suppose $A, B, C \in B(H)$ are invertible and $A \overset{s.s}{\sim} B$ and $B \overset{s.s}{\sim} C$. Then there exist invertible similitude operators $S_1, S_2 \in B(H)$ such that $A = S_1 B$ and $B = S_2 C$. Then
$A = S_1 B = S_1 S_2 C = SC$, where $S = S_1 S_2$ is a similitude by Theorem 2.1. This proves that $A \overset{s.s}{\sim} C$. Thus $\overset{s.s}{\sim}$ is an equivalence relation on the class of invertible operators.

3. Self-similarity and Similarity of Operators

Theorem 3.1 Let $A, B, S \in B(H)$. Then $B = SA$ if and only if $Ker(B) = Ker(A)$ (equivalently, $Ran(B) \subseteq Ran(S)$).

Proof
$Ker(B) = \{x: Bx = 0\}$
$= \{x: SAx = 0\}$
$= \{x: Ax = 0\}$
$= Ker(A)$.

Corollary 3.2 If $A, B \in B(H)$ are self-similar, then $Ker(A) = Ker(B)$.
Proof. Follows from Theorem 3.1.
Note that similarity of $A, B \in B(H)$ need not satisfy Corollary 3.2. The operators $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are similar, but $Ker(B) \neq Ker(A)$. A simple computation shows that these operators are similar but not self-similar. This shows that there exist similar operators which are not self-similar. We also note that while similarity preserves invertibility, self-similarity need not preserve invertibility. To see this, let $A = I$and $B = S$, the unilateral shift on $\ell^2(\mathbb{N})$. Clearly,A and $B$are self-similar but are not both invertible.

Theorem 3.3 If $A, B \in B(H)$ are similar and self-similar, then $A = B$.

Theorem 3.4 Self-similar operators are α −metrically equivalent.
Proof. Suppose that $A, B \in B(H)$ are such that $A = SB$for some similitude$S = \alpha V$, where $V$is an isometry and $\alpha > 0$. A simple computation gives $A^*A = \alpha^2 B^*B$, which establishes the claim.
We note that if $T \in B(H, K)$ is an α −isometry, then
$\langle Tx, Ty \rangle = \langle \alpha x, \alpha y \rangle = |\alpha|^2 \langle x, y \rangle$, for all $x \in H$and$y \in K$. In this case, we say that $T$is α −*isometric*.

Theorem 3.5 Two similitude operators on same space are metrically equivalent if and only if they have equal scaling factors.
Proof. Suppose that $A, B \in B(H)$ are such that $A = \alpha V$and $B = \beta W$, where V and $W$are isometries. Then $A^*A = \alpha^2 I = \beta^2 I = B^*B$ if $\alpha^2 = \beta^2$ and hence $\alpha = \beta$ by the strict positivity of both $\alpha$and β. The converse is trivial.
Theorem 3.5 can relaxed and extended to the following result.

Corollary 3.6 An α −isometry and a β −isometry are metrically equivalent if and only if the absolute value of their scaling factors coincide.
Proof. Suppose that $A, B \in B(H)$ are such that $A = \alpha V$and $B = \beta W$, where V and $W$are isometries and $\alpha, \beta \in \mathbb{C}$. Then $A^*A = |\alpha|^2 I = |\beta|^2 I = B^*B$ if $|\alpha|^2 = |\beta|^2$ and hence $|\alpha| = |\beta|$. The converse is trivial.
An operator $T \in B(H)$ is called an α −*unitary* if $T = \alpha U$, where $U$is a unitary operator and $0 \neq \alpha \in \mathbb{C}$.
Clearly, every unitary operator is an α −unitary and every α −unitary is an α −isometry. That is

$$Unitary \subset \alpha - unitary \subset \alpha - isometry$$

The converse is not true in general. The unilateral shift on $\ell^2(\mathbb{N})$ is an α −isometry which is not an α −unitary.

Theorem 3.7 Let $T \in B(H)$ be an α −isometry and $\alpha \neq 0$. Then $T$is normal if and only if T is invertible.
Proof. Suppose $T = \alpha V$, where Van isometry. Then
$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{11} \end{bmatrix}$, $T^*T = |\alpha|^2 I = |\alpha|^2 VV^* = TT^*$ if and only if $VV^* = I$if and only if $V$is a co-isometry. Thus $V$is unitary and hence invertible. This implies also implies the invertibility of T.

Corollary 3.8 An α −isometry is normal if and only if it is an α −unitary.

Corollary 3.9 An α −isometry is α −unitary if and only if it is invertible.

From Corollary 3.8 and Corollary 3.9, we conclude that every α −unitary is invertible and that every α −unitary is a normal operator.

Theorem 3.10 An operator $T \in B(H)$ is an α −unitary if and only if it is a normal α −isometry and $0 \neq \alpha \in \mathbb{C}$.
Proof. Suppose $T = \alpha U$, where Ua unitary operator. Then trivially, T is an normal α −isometry, by Corollary 3.8. Conversely, suppose that $T = \alpha U$, where Uan isometry and $0 \neq \alpha \in \mathbb{C}$. Normality of Timplies that $I - UU^* = 0$, which shows that $U$is a co-isometry, and hence a unitary operator. This proves the claim.
Theorem 2.2 can be sharpened as follows.

Theorem 3.11([3], Theorem 3.1) Quasisimilar isometries are unitarily equivalent.
Proof. See ([7], Theorem 1).
Theorem 3.11 can be extended as follows.

Theorem 3.12 Quasisimilar α −isometries are unitarily equivalent.
Proof. Suppose that $A = \alpha V$and $B = \alpha W$, where V and $W$are isometries and $\alpha \in \mathbb{C}$and let $X$and Ybe quasiaffinities such that $XA = BX$ and $AY = YB$. Upon substitution we have $XV = WX$and $VY = YB$. This shows that V and $W$are quasisimilar isometries and hence are unitarily equivalent, by Theorem 3.11. This also implies that A and B are unitarily equivalent.

Theorem 3.13 Let $T \in B(H)$ be an α −isometry. If $T$is invertible, then$\sigma(T) \subseteq \alpha\partial D$. If $T$is not invertible, then $\sigma(T) = \overline{\alpha D}$, where D denotes the open unit disk $\{z \in \mathbb{C}: |z| < 1\}$, and ∂Ddenotes the unit circle $\{z \in \mathbb{C}: |z| = 1\}$ and $\overline{D}$ denotes the open unit disk $\{z \in \mathbb{C}: |z| \leq 1\}$.

Lemma 3.14 Every isometry V is quasinormal.
Proof. Since $V$is an isometry, $V^*V = I$ and hence $V^*VV - VV^*V = 0$.
Lemma 3.14 can be generalized as follows.

Theorem 3.15 Every$\alpha -$ isometry $T$is quasinormal.
Proof. Suppose $T = \alpha V$, where Van isometry and $\alpha \in \mathbb{C}$. Without loss of generality, assume that $0 \neq \alpha \in \mathbb{C}$. Using Lemma 3.14 we have $(T^*T - TT^*)T = \alpha|\alpha|^2(V^*VV - VV^*V) = 0$.

4. Commutant of α −isometries

Let $T \in B(H)$. The commutant of Tdenoted by $\{T\}'$is the set of operators that commute with T. That is

$\{T\}' = \{A \in B(H): TA = AT\}$.

Theorem 4.1 The commutant of an isometry or co-isometry T on $\ell^2(\mathbb{N})$ consists of operators similar to scalar operators.
Proof. Without loss of generality, suppose $T$is a co-isometry. Then T has an infinite matrix representation of the form $T = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ with respect to the canonical basis in$\ell^2(\mathbb{N})$.
A simple computation shows that an operator $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \{T\}'$ if $A_{21} = 0$and $A_{11} = A_{22}$. That is, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{11} \end{bmatrix}$, which is similar to a scalar operator$S = A_{11}I$. This proves the claim.

Corollary 4.2 The commutant of an α −unitary operator T consists of isometries.

5. Similitudes and Geometry

Definition 5.1 Two sets are said to be *congruent* or *isometric (*respectively, *similar)* if one can be transformed into the other by an isometry (respectively, a similarity transformation).
Loosely speaking, an isometry is any geometric transformation that does not change the shape or size of a figure. This is a distortion less transformation. These include rotations, translations and reflections. A similitude is any transformation that changes the shape or size of a figure. This is a distortion transformation. Under this transformation, the two figures are similar.
Note that if $T \in B(H)$ is a similitude with scale factor α, and if Ω is a closed and bounded set in H, then the image $T(\Omega)$ of the set Ω under $T$is congruent to Ω scaled by α. In general, if $T: M \to N$is an isometry, then $Ran(T)$ is isometric/congruent to M. This can also be extended to $\alpha -$ isometries.

Definition 5.2 Two sets are said to be α −*congruent* or α −*isometric* if one can be transformed into the other by an $\alpha -$ isometry.

Proposition 5. 3 Let $T \in B(H, K)$ and $M \subseteq H$ and $N \subseteq K$. If $T: M \to N$is an α −isometry, then $Ran(T)$ is $\alpha -$ isometric/α −congruent to M.

We also note that similitudes and α −isometries need not preserve norm. Self-similarity is a notion of geometric similarity. This involves either distortions (affine transformations) or distortion-less linear operators.

Proposition 5.4 If two figures are similar with a scale factorα, then the perimeter ratio is $\alpha$and the area ratio is α^2.

Example 4. Suppose that in the Hilbert space $H = \mathbb{R}^2$, two closed and connected subspaces $\Omega_1$and Ω_2 are similar, with a scaling factor $\alpha > 0$. Then there exists a similitude$T: \Omega_1 \to \Omega_2$. A simple computation shows that the perimeter ratio is $\alpha$and the area ratio is α^2.

Remark. Example 4 can be extended toα −isometries (that is, all$\alpha \in \mathbb{C}$). In any case, the distance between points is $|\alpha|$. This is the ratio of magnification or dilation factor or similitude factor.

6. Discussion

Similitudes that are contractions (that is, $\boldsymbol{\alpha}$ −isometries where the scaling factor$0 < \alpha < 1$) are applicable in the study of fractals and are a popular tool to provide existence of solutions in many problems. Self-similarity is a characterization of fractals arising from iterated function systems and find application in chaotic dynamical systems and modelling of systems (see [10]). Similitudes in general reduce or increase distance between points and are very applicable in the study and application of frames, specifically generalized or tight g −frames ([1], Definition 3.1) and wavelets in image compression and decompression schemes, geometry and DNA sequences, gene mutation and fractals (see [8]).

7. Conclusion

In this paper, the notions of α −isometry and self-similarity are introduced. It has been shown that self-similarity is an equivalence relation. It has also been shown that quasisimilar α −isometries are unitarily equivalent. It is also shown that the commutant of an α − unitary operator T consists of isometries.

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