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Weyl and Browder theorems for operators with or without SVEP at zero

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Abstract
The study of operators having some special spectral properties like Weyl's theorem, Browder's theorem and the SVEP has been of important interest for some time now. The SVEP is very useful in the study of the local spectral theory. In this paper, we explore the single-valued extension property (SVEP) for some operators on Hilbert spaces. We characterize operators with or without SVEP at zero and those where Weyl's and Browder's theorems hold. It is shown that if a Fredholm operator has no SVEP at zero, then zero is an accumulation point of the spectrum of the operator. It is also shown that quasi similar Fredholm operators have equal Weyl spectrum.

Keywords: SVEP, quasi-similarity, ascent, descent, resolvent

1. Introduction

In this paper H will denote a complex separable Hilbert space and $B(H)$ will denote the Banach algebra of bounded linear operators. We denote by $G(H)$ the subalgebra of $B(H)$ of invertible operators. If $T \in B(H)$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H, respectively. We denote the cokernel of $T \in B(H)$ by $Ker(T^*) = H / Ran(T) = Ran(T)^{\perp}$. We denote by $\sigma(T), \|T\|, \partial\sigma(T)$ the spectrum, the norm of T and the boundary of the spectrum of T, respectively. We denote by $D, \overline{D}$ and ∂D the open unit disc, the closed unit disc and the boundary of the unit disc in $\mathbb{C}$, respectively

Two operators $A \in B(H)$ and $B \in B(K)$ are said to be *similar* if there exists an invertible operator $N \in B(H,K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are *unitarily equivalent* if there exists a unitary operator $U \in B_+(H,K)$ (Banach algebra of all invertible operators in $B(H)$) such that $UA = BU$ (i.e. $A = U^*BU$ equivalently, $A = U^{-1}BU$). An operator $X \in B(H,K)$ is a *quasiaffinity* or a *quasi-invertible* if it is injective and has dense range. Two operators $A \in B(H)$ and $B \in B(K)$ are said to be quasiaffine transforms of each other if there exists a quasi-affinity $X \in B(H,K)$ such that $XA = BX$.

Two operators $A \in B(H)$ and $B \in B(K)$ are *quasisimila*r if there exist quasiaffinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $AY = YB$.

A subspace $M \subseteq H$ is said to be *invariant* under $T \in B(H)$ if $TM \subseteq M$. In this case, we say that the subspace M is $T-$invariant.

A subspace $M \subseteq H$ is said to be a *reducing subspace* of $T \in B(H)$ if it is invariant under both T and T^* (equivalently, if both M and $M^{\perp}$ are invariant under T).

The spectrum of $T \in B(H)$ is given by $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not invertible $\}$. We denote by $\rho(T) := \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}$, and $Ran(\lambda I - T) = H\} = \mathbb{C} / \sigma(T)$, the resolvent of T. Equivalently, $\rho(T)$ is precisely the set $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is invertible. We denote by $\sigma_p(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) \neq \{0\}\}$ and $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \|\lambda I - T\| x_n \to 0, \|x_n\| = 1\} = \{\lambda \in \mathbb{C} : \lambda I - T$ is not bounded below$\}$, the point spectrum (set of all eigenvalues of T) and the approximate point spectrum of T, respectively. The approximate point spectrum of T is a non-empty closed subset of $\mathbb{C}$ and includes the boundary $\partial\sigma(T)$ of the spectrum of T. We denote by

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\} \text{ and } Ran(\lambda I - T) \neq \overline{Ran(\lambda I - T)} = H\}$$

the continuous spectrum of T, and we denote by $\sigma_r(T) = \{\lambda \in \mathbb{C} : Ker(\lambda I - T) = \{0\}$ and $\overline{Ran(\lambda I - T)} \neq H\}$ the residual spectrum of T. Clearly, $\sigma(T) = \sigma_P(T) \bigcup \sigma_c(T) \bigcup \sigma_r(T)$.

For any $T \in B(H)$ and any $n \in \mathbb{N} \bigcup \{0\}$, we have the ascending sequence

$$Ker(T^n) \subseteq Ker(T^{n+1}) \subseteq \cdots \bigcup_n Ker(T^n)$$

and the descending sequence

$$\bigcap_n Ran(T^n) \subseteq \cdots \subseteq Ran(T^{n+1}) \subseteq Ran(T^n).$$

The *ascent* $asc(T)$ of an operator T is the least $n \in \mathbb{N} \bigcup \{0\}$ such that $Ker(T^n) = Ker(T^{n+1})$ and *descent* $dsc(T)$ of T is the least $n \in \mathbb{N} \bigcup \{0\}$ such that $Ran(T^n) = Ran(T^{n+1})$. If no such n exists, then $asc(T) = \infty$ and $dsc(T) = \infty$, respectively. An operator $T \in B(H)$ *is Fredholm* if nullity $\alpha(T) = \dim(Ker(T)) < \infty$, $Ran(T)$ is closed and deficiency $\beta(T) = \dim(Ker(T^*)) < \infty$. We denote by $F(H)$ the set of Fredholm operators on a Hilbert space H. We define the index of a Fredholm operator T as $ind(T) = \alpha(T) - \beta(T)$. On a finite dimensional Hilbert space every operator is Fredholm with index 0.

A point $\lambda \in \sigma(T)$ is said to have *finite multiplicity* if nullity $\alpha(\lambda I - T) < \infty$. If K is a set, we write $iso\ K = K \setminus acc\ K$, where $iso\ K$ is the set of isolated points of K and $acc\ K$ is the set of accumulation points of K. By definition, an accumulation point of a set K is a limit point of K. It can lie in the set K or can be a boundary point of K. Clearly every isolated point of a set K is in a boundary point of K.

We denote by $iso\ \sigma(T)$ the isolated point spectrum of T (i.e. the set of isolated eigenvalues of T of finite multiplicity). Clearly $iso\ \sigma(T) \subseteq \partial\sigma(T)$. A complex number $\lambda_0 \in iso\ \sigma(T)$ is a *pole* of the resolvent. of T if and only if $0 < asc(\lambda I - T) = dsc(\lambda I - T) < \infty$. This is a necessary and sufficient condition for λ_0 to be a pole of the the resolvent. of T. In this case, the positive integer $p = asc(\lambda I - T)$ is the *order* of the pole λ_0 and the point λ_0 is an isolated point of $\sigma(T)$. A complex number λ_0 is called a *Riesz.point* of T if $0 < asc(\lambda I - T) = dsc(\lambda I - T) < \infty$ and nullity $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$. Note that a Riesz point of T is either a pole of the resolvent (and hence an isolated point of $\sigma(T)$) or a point in the resolvent of T . We will denote the set of Riesz points of T by $\pi_0(T)$.

By $\pi_0(T)$, $\pi_{00}(T)$, and $\pi_{00}{}^a(T)$ we denote the set of poles of the resolvent of T of finite multiplicity, the set of isolated eigenvalues of T of finite multiplicity and the set of isolated points of $\sigma_{ap}(T)$ which are eigenvalues of finite multiplicity, respectively. Clearly,

$$\pi_{00}(T) = \{\lambda \in iso\ \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$\pi_0(T) \subseteq \pi_{00}(T) \subseteq \sigma_P(T) \text{ and } \pi_{00}{}^a(T) \subseteq \sigma_{ap}(T).$$

From the definitions, we have that a necessary and sufficient condition for $\lambda_0 \in \pi_0(T)$ is that $asc(\lambda I - T) = dsc(\lambda I - T) < \infty$. It is also clear that $\pi_0(T) = \{\lambda \in \pi_{00}(T) : \lambda I - T$ is Fredholm$\}$.
It is well known (see [16], Theorem 3) that if $asc(\lambda I - T)$ and $dsc(\lambda I - T)$ are finite, then they are equal, and their common value is the order of the pole at λ.

We define the essential spectrum of T by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\} = \{\lambda \in \mathbb{C} : Ran(\lambda I - T) \text{ is not closed}, \alpha(\lambda I - T) = \infty \text{ or } \beta(\lambda I - T) = \infty\}.$$

Clearly, $\sigma_e(T)$ is a compact subset of $\mathbb{C}$ and it is empty if H is finite-dimensional (see [12], Remark 4.3). It is known that $\sigma_e(T) = \sigma_e(T^*)$ for any operator T (see [14], Proposition 2.6.1).
An operator $T \in B(H)$ is called a *Weyl operator* if it is a Fredholm operator of index 0. We denote the class of Weyl operators by $W(H)$. That is, $W(H) = \{T \text{ Fredholm} : ind(T) = 0\}$.

We define the Weyl spectrum of T as

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\} = \{\lambda \in \mathbb{C} : ind(\lambda I - T) \neq 0\} = \bigcap_{K \sim compact} \sigma(T + K),$$

which is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations. Clearly $\sigma_w(T)$ is a compact subset of $\mathbb{C}$, because it is the intersection of compact sets. It is known that if H is finite-dimensional, then $\sigma_w(T) = \varnothing$ and if H is infinite-dimensional and T is compact, then $\sigma_w(T) = \sigma_e(T) = \{0\}$ (see [12], Remark 7.3). Note that every operator on a finite dimensional Hilbert space is Weyl. An operator $T \in B(H)$ satisfies *Weyl's theorem* if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. An operator $T \in B(H)$ is said to be *Browde*r if it is Fredholm with finite ascent and finite descent. We denote the class of all Browder operators by $B_b(H)$. That is, $B_b(H) = \{T \text{ Fredholm} : asc(T) < \infty \text{ and } dsc(T) < \infty\}$.

We define the *Browder spectrum* of T as

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\} = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}, asc(\lambda I - T) = dsc(\lambda I - T) < \infty\}$$
$$= \bigcap_{\substack{K \sim compact \\ K \in \{T\}'}} \sigma(T + K),$$

where $\{T\}'$ denotes the commutant of T. From the definition, it is clear that $\sigma_b(T)$ is the largest part of $\sigma(T)$ that remains unchanged under compact perturbations in the commutant of T. Clearly, every invertible operator as well as well as every Browder operator is a Weyl operator. In fact

$$G(H) \subset B_b(H) \subset W(H) \subset F(H).$$

Evidently, $\sigma_b(T)$ is a compact subset of $\sigma(T)$ and is non-empty if the Hilbert space H is infinite-dimensional (see [12], Remark 9.3) and empty in finite dimensional Hilbert spaces. It is known that

$$\sigma_e(T) \subset \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \bigcup acc\ \sigma(T).$$

In finite dimensions, we have

$$\sigma_e(T) = \sigma_w(T) = \sigma_b(T) = \varnothing.$$

The complement in $\sigma(T)$ of the Browder spectrum is denoted by

$$P_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\}.$$

This set coincides with $\pi_0(T)$ the set of all poles of the resolvent of T having finite multiplicity (also called the set of all Riesz

points in $\sigma(T)$).

Clearly, $P_{00}(T) \subseteq \pi_0(T)$ for every $T \in B(H)$. In fact

$$iso\ \sigma(T) \setminus \sigma_e(T) = iso\ \sigma(T) \setminus \sigma_w(T) = P_{00}(T) \subseteq \pi_{00}(T).$$

An operator $T \in B(H)$ satisfies *Browder's theorem* if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ or equivalently, $\sigma(T) \setminus \sigma_w(T) = P_{00}(T)$. This is equivalent to saying that $\sigma_w(T) = \sigma_b(T)$. This concept was introduced in 1997 by Harte and Lee [1]. It is known that for any $T \in B(H)$:

$$Weyl's\ theorem \Rightarrow Browder's\ theorem.$$

The investigation of operators satisfying Weyl's theorem was initiated by H. Weyl[19] in 1909 (see also [18]) who studied the spectra of all compact perturbations $T + K$ of a Hermitian operator T on a Hilbert space H and showed that $\lambda \in \mathbb{C}$ belongs to $\sigma(T + K)$ for every compact operator K precisely when λ is not an isolated point of finite multiplicity in $\sigma(T)$. Weyl[19] proved that for every Hermitian operator T on a complex Hilbert space H, we have $\sigma_w(T) = \sigma(T) \setminus \pi_{00}(T)$. This remarkable result describing the largest subset of the spectrum remaining invariant under arbitrary compact perturbation was extended by [18] to several classes of operators including $p-$hyponormal and $M-$hyponormal operators.

An operator $T \in B(H)$ is said to be *isoloid* if every isolated part of $\sigma(T)$ is an eigenvalue of T. This is equivalent to saying that if $\lambda \in iso\ \sigma(T)$ then $\lambda \in \sigma_p(T)$.

That is, $iso\ \sigma(T) \subseteq \sigma_p(T)$. This is also equivalent to saying that every isolated point of $\sigma(T)$ is in $\pi_{00}(T)$.

An operator T is said to be *polaroid* if every isolated point of $\sigma(T)$ is a pole of the resolvent set $\rho(T)$ of T. That is $iso\ \sigma(T) \subseteq \pi_0(T)$. Clearly every polaroid operator is isoloid, but the converse is not true in general.

An operator T is said to be *decomposable* if for any open covering $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$, there are two closed $T-$invariant subspaces M_1 and M_2 of H such that $M_1 + M_2 = H$ and $\sigma(T|_{M_k}) \subseteq U_k$, for $k = 1, 2$.

Let H be a Hilbert space and $U \subseteq C$ be an open set. A map $f : U \to H$ is said to be analytic if for every $\lambda_0 \in U$ there is $r > 0$ and $a_k \in H$ such that $f(\lambda) = \sum_{k=0}^{\infty} a_k\ (\lambda - \lambda_0)^k$ for all $\lambda \in U$ with $|\lambda - \lambda_0| < r$.

2. The SVEP on a Subset of $\mathbb{C}$, Weyl's and Browder's Theorems

Background of the SVEP: Let $T \in B(H)$ and fix $0 \neq x \in H$. The function $f(\lambda) := (\lambda I - T)^{-1} x$ is analytic on the resolvent set $\rho(T)$ of T. A function $\hat{f}$ is an analytic extension of f if it is analytic and $\hat{f}(z) = f(z)$ for all $z \in \rho(T)$ and that $(\lambda I - T)\hat{f}(\lambda) = x$, for all $x \in Dom(f)$. If for all $0 \neq x \in H$, any two extensions $\hat{f}_1, \hat{f}_2$ of $f(\lambda) := (\lambda I - T)^{-1} x$ agree on all $x \in Dom(\hat{f}_1) \bigcap Dom(\hat{f}_2)$, we say that T has the *single-valued extension property*, abbreviated SVEP.

An operator $T \in B(H)$ is said to have the single-valued extension property (SVEP) at a point λ_0 if for every non-empty open set $U \subseteq \mathbb{C}$ containing or centred at λ_0 the only analytic solution $f : U \to H$ of the equation

$$(\lambda I - T) f(\lambda) = 0,\ \lambda \in U$$

is the zero function, $f \equiv 0$ (that is f is identically zero). An operator T is said to have the SVEP if it has the SVEP at every $\lambda \in \mathbb{C}$. We denote by $O(U, H)$ the Frechet space of $H-$valued analytic functions on the open subset U of $\mathbb{C}$, equipped with the topology of uniform convergence on compact subsets of U. Every $T \in B(H)$ induces a continuous mapping T_U on $O(U, H)$ by

$$T_U(f)(\lambda) = (\lambda I - T) f(\lambda),$$

for all $f \in O(U, H)$. The operator T has SVEP if T_U is injective.

Evidently, $T \in B(H)$ has the SVEP at every point of the resolvent. Moreover from the identity theorem for analytic functions it is easily seen that T has the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has the SVEP at every isolated point of the spectrum that is $\lambda \in iso\ \sigma(T)$.

An operator $T \in B(H)$ is said to have the *Bishop's property* (β) if $f_n(\lambda)$ is an analytic vector-valued function on some open set U such that $(\lambda I - T) f_n(\lambda) \to 0$ uniformly on each compact set $V \subseteq U$, V then $f_n(\lambda) \to 0$ uniformly on V. Clearly, if

T has Bishop's property (β) then T has the SVEP.

The SVEP was introduced by Dunford (see [6], [7]) and it plays an important role in local spectral theory and Fredholm theory, especially Weyl's theorem of operators and its generalizations. Note every operator has SVEP. Finch[8] has proved that if an operator is surjective but not injective, then it does not have SVEP. It has been shown in [10] that $p-$hyponormal, $M-$hyponormal and log-hyponormal operators possess the single valued extension property, as does any operator with empty point spectrum. This result has been extended to some higher classes of non-normal operators. For instance, Duggal and Kubrusly ([5], Corollary 2.4) have shown that the class of $k-$paranormal operators has SVEP.

Theorem 2.1: Let H and K be Hilbert spaces. If $A \in B(H)$ has the SVEP at $\lambda_0 \in \mathbb{C}$ and $B \in B(K)$ is a quasiaffine transform of A, then B has the SVEP at λ_0.

Proof: Suppose that $X \in B(H,K)$ is a quasiaffinity such that $AX = XB$. Let $f : U \to K$ be an analytic function defined on an open neighbourhood U of λ_0 such that $(\lambda I - B) f(\lambda) = 0$ for all $\lambda \in U$. Then $X(\lambda I - B) f(\lambda) = (\lambda I - A) X f(\lambda) = 0$. Using the SVEP of A at λ_0, we conclude that $f(\lambda) = 0$ for all $\lambda \in U$. This proves that B has the SVEP.

Theorem 2.1 says that the SVEP is stable under quasiaffine transformation. Theorem 2.1 can be strengthened as follows.

Theorem 2.2: H and K be Hilbert spaces. Suppose $A \in B(H)$ and $B \in B(K)$ are quasisimilar,. Then A has the SVEP at λ_0 if and only if B has the SVEP at λ_0.

Proof: One direction has been proved in Theorem 2.1. We prove the converse. Suppose B has the SVEP and $(\mu I - A) g(\mu) = 0$, where $g : U \to H$ is analytic on U. Using the proof of Theorem 2.1 and $YA = BY$ for a quasiaffinity Y, we have that A has the SVEP.

Theorem 2.2 says that the SVEP is stable under quasisimilarity. Since unitary equivalence and similarity imply quasisimilarity, we conclude also that the SVEP is invariant under unitary equivalence and similarity of operators.
We need the following basic results.

Lemma 2.3: If $\lambda \in iso\ \sigma(T)$, then $\lambda \in \sigma_p(T)$.

If T is compact, then $\sigma(T)$ is a countable set and has no accumulation point except possibly 0. In this case, every non-zero number in $\sigma(T)$ is an eigenvalue of T and also a pole of the resolvent of T.

Lemma 2.4: If λ is an accumulation point of $\sigma(T)$ or an eigenvalue of infinite multiplicity, then $\lambda \in \sigma_e(T)$.

Proof: We prove the first claim by contradiction. Suppose that $\lambda \notin acc\ \sigma(T)$. Then $\lambda \in iso\ \sigma(T)$. Thus λ is a pole of the resolvent of T and $\dim(\{P_{\{\lambda\}}\}) < \infty$. Thus $\lambda \in \pi_0(T)$ and therefore $\lambda \in \sigma_e(T)$. On the other hand, if λ is an eigenvalue of infinite multiplicity, then $\alpha(\lambda I - T) = \infty$. By definition this means that $\lambda \in \sigma_e(T)$.

It is well known that quasisimilar operators need not have the same spectrum. The following result shows that if at least one of these operators has an isolated spectrum and has the SVEP, then they have the same spectrum.

Theorem 2.5: Suppose that H and K be Hilbert spaces and suppose $A \in B(H)$ and $B \in B(K)$ are quasisimilar. If A A or B has the SVEP at λ_0 with an isolated spectrum, then $\sigma(A) = \sigma(B)$.

Proof: Since A and B are quasisimilar, then $\lambda I - A$ and $\lambda I - B$ are also quasisimilar. Without loss of generality, suppose A has the SVEP at λ_0 and has isolated spectrum. Then by Theorem 2.2, B has the SVEP at λ_0. This is equivalent to saying that $\lambda_0 I - A$ is injective implies that $\lambda_0 I - B$ is injective. That is $\sigma(A) \subseteq \sigma(B)$. By symmetry, we have $\sigma(B) \subseteq \sigma(A)$. Combining these two statements, we establish the claim.

Proposition 2.6: Let $T \in B(H)$ and let $\lambda_0 \in U \subseteq \mathbb{C}$, where U is an open set. If $\lambda_0 I - T$ is injective, then T has the SVEP at λ_0.

Proof: Suppose that $f : U \to H$ is an analytic function defined on an open neighbourhood U of λ_0 such that $(\lambda I - T) f(\lambda) = 0$ for all $\lambda \in U$. Since $\lambda I - T$ is injective, we conclude that $f(\lambda) = 0$ for all $\lambda \in U$. This proves that T has the SVEP at λ_0.

Proposition 2.6 asserts that any operator T has the SVEP on its resolvent set $\rho(T)$. A consequence of Proposition 2.6 is the following result.

Corollary 2.7: Let $T \in B(H)$. If T is injective, then T has the SVEP at 0.

Proof: Without loss of generality, we let $\lambda_0 = 0$. The rest of the proof then follows from Proposition 2.6.

The following result follows is a consequence of Finch [8] .

Proposition 2.8: Let $T \in B(H)$ and let $\lambda_0 \in U \subseteq \mathbb{C}$, where U an open set. If $\lambda_0 I - T$ is surjective, then T has the SVEP at λ_0 if and only if $\lambda_0 \in \rho(T)$.

The following is a consequence of Proposition 2.8.

Corollary 2.9: Let $T \in B(H)$ and let $\lambda_0 \in U \subseteq \mathbb{C}$, where U an open set is. If $\lambda_0 I - T$ is surjective, then T has the SVEP at λ_0 if and only if $\lambda_0 I - T$ is injective.

Proof. Without loss of generality, we let $\lambda_0 = 0$. Assume T is surjective and has SVEP at 0. That is $Ran(T) = H$ and so $Ker(T) \bigcap H = \{0\}$, so T is injective. The converse follows from Proposition 2.8.

P. Aiena[1] has suggested a way to obtain operators that do not have the SVEP using Corollary 2.7 and has shown that for an operator $T \in B(H)$ there exists a closed subspace M of H such that $(\lambda_0 I - T)M = M$ and $Ker(\lambda_0 I - T) \bigcap M \neq \{0\}$, then T does not have the SVEP at λ_0. Using this result, we are able to find operators without SVEP at 0.

We note that T has SVEP does not in general imply that T^* has SVEP. Recall that an operator is said to be quasi-nilpotent if $\sigma(T) = \{0\}$.

Theorem 2.10: If T is quasi-nilpotent then T^* has the SVEP.

Proof: T quasi-nilpotent implies that T has the SVEP and T and T^* are quasisimilar. By Theorem 2.2 we conclude that T^* has the SVEP.

Example: Let T be the unilateral shift on the Hilbert space $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma_w(T) = \overline{D}$. This operator has no eigenvalues, that is $\sigma_p(T) = \varnothing$ and hence $\pi_{00}(T) = \pi_0(T) = \varnothing$. Thus T satisfies Weyl's theorem. It is also clear that T has the SVEP. However, T^* does not have the SVEP.

Theorem 2.11: If T satisfies Weyl's theorem then Browder's theorem holds for T and $\sigma(T) = \sigma_w(T) \bigcup iso\ \sigma(T)$.

Proof: Since T satisfies Weyl's theorem, to prove that Browder's theorem holds for T it is sufficient to prove that T has SVEP at every $\lambda \notin \sigma_w(T)$ ([2], Theorem 3.1). Suppose $\lambda \notin \sigma_w(T)$. We consider two cases:

If $\lambda \notin \sigma(T)$, then $\lambda \in \rho(T)$ and therefore $\lambda I - T$ is invertible and so has the SVEP at λ. On the other hand, if $\lambda \in \sigma(T)$ and T satisfies Weyl's theorem, then $\lambda \in \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Thus $\lambda \in iso\ \sigma(T)$, which implies that T has SVEP at λ. This proves that Browder's theorem holds for T. The last claim follows from the fact that $\lambda \in \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) \subseteq iso\ \sigma(T)$. Thus $\sigma(T) \subseteq \sigma_w(T) \bigcup iso\ \sigma(T)$. But the reverse inclusion $\sigma_w(T) \bigcup iso\ \sigma(T) \subseteq \sigma(T)$ is obvious. This proves our last claim.

The following result gives a necessary and sufficient condition for Browder's theorem to hold.

Corollary 2.12: Browder's theorem holds for T if and only if $acc\ \sigma(T) \subseteq \sigma_w(T)$.

Proof. Recall that for any operator $T \in B(H)$ we have that $\pi_0(T) \subseteq iso\ \sigma(T)$. We first note that

$$iso\ \sigma(T) \setminus \sigma_e(T) = iso\ \sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq \pi_{00}(T).$$

If Browder's theorem holds for T then

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq iso\ \sigma(T).$$

Taking complements both sides, we get

$$acc\ \sigma(T) \subseteq \sigma_w(T).$$

Conversely, suppose that $acc\ \sigma(T) \subseteq \sigma_w(T)$. Taking complements both sides of the inclusion, we have that $\sigma(T) \setminus \sigma_w(T) \subseteq iso\ \sigma(T)$. Subtracting $\sigma_w(T)$ both sides we have

$$(\sigma(T) \setminus \sigma_w(T)) \setminus \sigma_w(T) = \sigma(T) \setminus \sigma_w(T) = iso\ \sigma(T) \setminus \sigma_w(T) = \pi_0(T).$$

This shows that T satisfies Browder's theorem.

Theorem 2.12: Let $T \in B(H)$. If λ_0 is an isolated point of $\sigma_p(T)$ then T has SVEP at λ_0.

Proof: Since λ_0 is not an accumulation point of $\sigma_p(T)$, there is an open set U containing λ_0 such that the operator $\lambda I - T$ is injective for every $\lambda \in U$, $\lambda \neq \lambda_0$. Let V be another open set containing λ_0 and suppose that $f : V \to H$ is an analytic function for which the equation $(\lambda I - T) f(\lambda) = 0$ for all $\lambda \in V$. Without loss of generality, we may assume that $U \subseteq V$. Then the injectivity of $\lambda I - T$ implies that $f(\lambda) \in Ker(\lambda I - T) = \{0\}$ for every $\lambda \in V$, $\lambda \neq \lambda_0$. This says that $f(\lambda) = 0$ for ever $\lambda \in V$, $\lambda \neq \lambda_0$. Since f is continuous at λ_0, we conclude that $f(\lambda_0) = 0$. Hence $f \equiv 0$ in V and therefore T has SVEP at λ_0.

Theorem 2.13: Let $T \in B(H)$. Then T has SVEP on $\rho(T)$.

Proof: Recall that $\rho(T)$ is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda I - T$ is bijective on H. clearly bijectivity of $\lambda I - T$ implies that $Ker(\lambda I - T) = \{0\}$ and $Ran(\lambda I - T) = H$, which implies that $\lambda I - T$ is invertible. Hence by Proposition 2.6, $\lambda I - T$ has the SVEP on $\rho(T)$.

Theorem 2.14: Let $T \in B(H)$. Then T has SVEP on $\partial\sigma(T)$.

Proof: We first note that if $\lambda \in \partial\sigma(T)$, then the nullity $\alpha(\lambda I - T) < \infty$. Thus $\lambda \in \pi_0(T)$. Since $\sigma(T)$ is closed, we have that $\partial\sigma(T) \subseteq \sigma(T)$. Thus $\lambda \in \sigma(T)$ and is a point of finite multiplicity. This shows that $\lambda \in iso\ \sigma(T)$, and hence has the SVEP on $\partial\sigma(T)$.

All non-invertible surjective operators acting on infinite dimensional Hilbert spaces lack SVEP.
From Corollary 2.7, we give the following Fredholm Alternative type theorem.

Theorem 2.15: If $T \in B(H)$ is compact and $\lambda \neq 0$, then $\lambda I - T$ is injective if and only if it is surjective.

Corollary 2.16: ([14], Proposition 1.2.10) If $T \in B(H)$ is surjective and has the SVEP, then it is invertible.

Proposition 2.17: Every $T \in B(H)$ has SVEP on $\sigma_r(T)$.

Proof: The proof follows from the fact that $\lambda \in \sigma_r(T)$ if $\lambda I - T$ is injective but $\overline{Ran(\lambda I - T)} \neq H$. By Proposition 2.6, T has the SVEP on $\sigma_r(T)$.

Proposition 2.18: Every $T \in B(H)$ has SVEP on $\sigma_c(T)$.

Proof: The proof follows from the fact that $\lambda \in \sigma_c(T)$ if $\lambda I - T$ is injective but $Ran(\lambda I - T) \neq \overline{Ran(\lambda I - T)} = H$. By Proposition 2.6, T has the SVEP on $\sigma_r(T)$.

We now characterize Weyl's and Browder's theorems in terms of the point spectrum of an operator.

Theorem 2.19: Let $T \in B(H)$. If Weyl's theorem holds for T then $\sigma(T) \setminus \sigma_p(T) \subseteq \sigma_w(T)$.

Proof: From the inclusion

$$\pi_0(T) \subseteq \pi_{00}(T) \subseteq iso\ \sigma(T) \subseteq \sigma_p(T),$$

and taking complements and using the fact that $\{\sigma_p(T), \sigma_c(T), \sigma_r(T)\}$ forms a partition of $\sigma(T)$, and by taking complements in the inclusion $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, we have

$$\sigma_w(T) \supseteq \sigma_c(T) \bigcup \sigma_r(T) = \sigma(T) \setminus \sigma_p(T).$$

Theorem 2.20: Let $T \in B(H)$. If Browder's theorem holds for T then $\sigma(T) \setminus \sigma_p(T) \subseteq \sigma_b(T)$.

Theorem 2.21: Let $T \in B(H)$. If $Ker(T) \bigcap Ran(T) = \{0\}$, then T has SVEP at 0.

Proof: $Ker(T) \bigcap Ran(T) = \{0\}$ implies $H = Ker(T) \oplus Ran(T)$. Thus the subspaces $Ker(T)$ and $Ran(T)$ reduce T and hence $T = T_1 \oplus T_2$. This means that
T is decomposable and hence $\text{int}(\sigma_p(T)) = \varnothing$. Therefore T has SVEP.

Remark: The converse of Theorem 2.21 also holds for injective operators. If T has the SVEP at 0 then 0 either belongs to $\rho(T), iso\ \sigma(T), \partial\sigma(T)$ or T is injective.
An operator $T \in B(H)$ is said to be *isoloid* if every isolated point of the spectrum of T is an eigenvalue of T. That is if $iso\ \sigma(T) \subseteq \sigma_p(T)$.

Theorem 2.22: Let $T \in B(H)$ be isoloid. Then $\sigma_r(T) \bigcup \sigma_c(T) \subseteq acc\ \sigma(T)$.

Proof: By definition $\sigma(T) \setminus \sigma_p(T) = \sigma_r(T) \bigcup \sigma_c$. Since T is isoloid, we have $iso\ \sigma(T) \subseteq \sigma_p(T)$. Using the fact that $acc\ \sigma(T) = \sigma(T) \setminus iso\ \sigma(T)$, we have

$$\sigma_r(T) \bigcup \sigma_c(T) = \sigma(T) \setminus \sigma_p(T) \subseteq \sigma(T) \setminus iso\ \sigma(T) = acc\ \sigma(T).$$

3. Ascent, Descent, Index, Isolated points of the Spectrum, Poles of the Resolvent and the SVEP

It is also known (see [1], Theorem 3.1) that if both ascent and descent of T are finite, then they are equal.

Proposition 3.1: Let $T \in B(H)$. Then $asc(T) = 0$ if and only if T is injective and $dsc(T) = 0$ if and only if T is surjective.

Proof: Clearly, $asc(T) = 0$ is equivalent to saying that $Ker(T^0) = Ker(I) = \{0\} = Ker(T)$. This equivalent to saying that T is injective.
On the other hand, $dsc(T) = 0$ is equivalent to saying that $Ran(T^0) = Ran(T) = H$. This equivalent to saying that T is surjective.

Theorem 3.2: ([1], Theorem 3.8) for $T \in B(H)$ and $\lambda_0 \in \mathbb{C}$, the following assertions hold.

i) If $asc(\lambda_0 I - T) < \infty$ then T has the SVEP at λ_0.

ii) If $dsc(\lambda_0 I - T) < \infty$ then T^* has the SVEP at λ_0.

The following results show the connection between isolated points and the SVEP.

Lemma 3.3 ([16], Proposition 3)**:** Suppose T has the SVEP at $\lambda_0 = 0$ and $dsc(T) < \infty$. Then $asc(T) = dsc(T)$.

Lemma 3.4: Let $T \in B(H)$. Then the following assertions are equivalent.
(i). 0 is a pole of the resolvent of T.
(ii). T has the SVEP at 0 and $dsc(T) < \infty$.

Proof: (i) $\Rightarrow$ (ii): Suppose 0 is a pole of the resolvent of T. Then 0 is an isolated point in $\sigma(T)$, and so T has the SVEP at 0. Using the definition of a pole of the resolvent, we conclude that $dsc(T) < \infty$.
. (ii) $\Rightarrow$ (i): Suppose that T has the SVEP at 0 and $dsc(T) < \infty$. Then by Lemma 3.3, we have $0 < asc(T) = dsc(T) < \infty$. This shows that $\lambda = 0$ is a pole of the resolvent.

Theorem 3.5: Let $T \in B(H)$. Then the following assertions are equivalent.
(i). Weyl's theorem holds for T.
(ii). T satisfies Browder's theorem and $\pi_0(T) = \pi_{00}(T)$.

Proof: (i) $\Rightarrow$ (ii): Weyl's theorem implies Browder's theorem was proved in Theorem 2.11.
(ii) $\Rightarrow$ (i): Suppose Browder's theorem and $\pi_0(T) = \pi_{00}(T)$. Then T has SVEP and $\sigma(T) \setminus \sigma_w(T) = \pi_0(T) \subseteq \pi_{00}(T)$. The condition that $\pi_0(T) = \pi_{00}(T)$ then implies that $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. This proves the claim.
Note that Browder's theorem alone is not enough for Weyl's theorem to hold for any $T \in B(H)$. However, if T is compact then Browder's theorem implies Weyl's theorem. Using ([11], Theorem 2) and the fact that $\sigma(T) \setminus \sigma_b(T) = \pi_0(T)$, we have that $\sigma(T) = \sigma_w(T) \bigcup \pi_{00}(T)$ and so $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$.

Theorem 3.6: Each of the following assertions are equivalent to Browder's theorem for T.
(i). $\pi_{00}(T) \subseteq \pi_0(T)$.
(ii). $\pi_{00}(T) \subseteq iso\ \sigma(T)$.
(iii). $\pi_{00}(T) \subseteq \partial\sigma(T)$.
(iv). $\text{int}(\pi_{00}(T)) = \varnothing$.

Proof: (i) $\Rightarrow$ (ii): The inclusion $\pi_0(T) \subseteq \pi_{00}(T)$ holds for every $T \in B(H)$. This together with (i) implies that $\pi_0(T) = \pi_{00}(T)$. Since $\pi_{00}(T) \subseteq iso\ \sigma(T) \subseteq \partial\sigma(T)$, we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).
Since $\text{int}(\partial\sigma(T)) = \varnothing$, we have that (iii) $\Rightarrow$ (iv). To prove the implication (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii), we use the fact that

since by the Punctured neighbourhood theorem we have

$$\partial\sigma(T) \subseteq \sigma_w(T) \bigcup iso\ \sigma(T).$$

Since $\pi_0(T) = \pi_{00}(T) \bigcap iso\ \sigma(T)$, we conclude that (ii) $\Rightarrow$ (iii).

A necessary and sufficient condition for T to satisfy Browder's theorem is that T has SVEP at $\lambda \notin \sigma_w(T)$. This implies that if T has SVEP $\lambda \notin \sigma_w(T)$ then $\sigma_b(T) = \sigma_w(T)$. Note that SVEP alone is not enough for an operator to satisfy Weyl's theorem. A necessary and sufficient condition for Weyl's theorem is that $\sigma_w(T) = acc\ \sigma(T)$ and $\pi_0(T) = \pi_{00}(T)$.

Proposition 3.7: For any $T \in B(H)$ we have that $\sigma_b(T) = \sigma_e(T) \bigcup acc\ \sigma(T) = \sigma_w(T) \bigcup acc\ \sigma(T)$.

Proof": Clearly

$$\begin{aligned}\sigma_w(T) &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_e(T) \text{ or } ind(\lambda I - T) \neq 0\} \\ &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_e(T) \text{ or } ind(\lambda I - T) < 0 \text{ or } ind(\lambda I - T) > 0\}. \\ &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_e(T) \text{ or } \alpha(\lambda I - T) < \beta(\lambda I - T) \text{ or } \alpha(\lambda I - T) > \beta(\lambda I - T)\} \\ &= \sigma_e(T) \bigcup acc\ \sigma(T).\end{aligned}$$

From ([12], Corollary 9.2), we have

$$\begin{aligned}\sigma_b(T) &= \{\lambda \in \mathbb{C} : \lambda \in \sigma_e(T) \text{ or } asc(\lambda I - T) \neq dsc(\lambda I - T)\} \\ &= \sigma_e(T) \bigcup acc\ \sigma(T).\end{aligned}$$

Proposition 3.8: ([13], Corollary 5.38) For any $T \in B(H)$, $\sigma_b(T) = \sigma(T) \setminus \pi_0(T)$.

Proposition 3.8 says that for any $T \in B(H)$ we have $\sigma(T) = \sigma_b(T) \bigcup \pi_0(T)$. Thus for any $T \in B(H)$, $\{\sigma_b(T), \pi_0(T)\}$ is a partition of $\sigma(T)$ in terms of the Browder spectrum.

Proposition 3.9 ([13], Corollary 5.39) For any $T \in B(H)$, $\pi_0(T) = \pi_{00}(T) \setminus \sigma_e(T)$.

Note that in finite dimensional Hilbert spaces, $\sigma_e(T) = \varnothing$. Thus by Proposition 3.9 we conclude that in finite dimensional spaces, $\pi_0(T) = \pi_{00}(T)$. However, this condition is also enjoyed by some operators acting on infinite dimensional Hilbert spaces.

Theorem 3.10 A necessary and sufficient condition for Weyl's theorem is that $\sigma_w(T) = acc\ \sigma(T)$ and $\pi_0(T) = \pi_{00}(T)$.
Proof: If $\sigma_w(T) = acc\ \sigma(T)$, then by Proposition 3.7, $\sigma_b(T) = \sigma_w(T)$. This means that T satisfies Browder's theorem. In addition, if $\pi_0(T) = \pi_{00}(T)$ then $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, which establishes the claim.
Note that if $\sigma_p(T) = \varnothing$, then $\pi_0(T) = \pi_{00}(T) = \varnothing$. Thus every operator with empty point spectrum satisfies Weyl's theorem and hence Browder's theorem and hence has the SVEP.

Example 1: Let $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots).$$

Then

$$\sigma(T) = \{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\},\ iso\ \sigma(T) = \{\frac{1}{n} : n \in \mathbb{N}\},\ acc\ \sigma(T) = \{0\}.$$

Also

$$\pi_{00}(T) = \{\frac{1}{n} : n \in \mathbb{N}\},\ \pi_0(T) = \{\frac{1}{n} : n \in \mathbb{N}\},\ \sigma_e(T) = \sigma_w(T) = \sigma_b(T) = \sigma(T) \setminus \{\frac{1}{n} : n \in \mathbb{N}\} = \{0\}$$

and

$$\sigma_p(T) = \{\frac{1}{n} : n \in \mathbb{N}\}.$$

Consequently,

$$\partial\sigma(T) = \sigma(T) - \mathrm{int}(\sigma(T)) = \sigma(T) = \{\frac{1}{n} : n \in \mathbb{N}\} \bigcup \{0\}.$$

Clearly T is compact, normal and hence satisfies Weyl's theorem and hence Browder's theorem. For this operator, $\mathrm{int}(\sigma_p(T)) = \varnothing$ and so T has the SVEP. Here, $\sigma(T)$ is at most countable.

Recall that a subset M of X is *dense* if $\overline{M} = X$. A subset M of X is *nowhere dense* if $\mathrm{int}(\overline{M}) = \varnothing$. A set M is said to be *somewhere dense* if it is not nowhere dense. We note that in Example 1, the spectrum of T is discrete, and hence nowhere dense in $\mathbb{C}$.

Theorem 3.11: Let $T \in B(H)$. If $\sigma(T)$ is nowhere dense then T has SVEP on $\sigma(T)$.

Proof: $\sigma(T)$ nowhere dense implies that $\sigma(T)$ is discrete and hence every $\lambda \in \sigma(T)$ is isolated. Thus $\sigma(T) = iso\ \sigma(T)$, and therefore T has SVEP on $\sigma(T)$. This proves the claim.

Remark: We note that Theorem 3.11 gives a necessary but not sufficient condition when an operator T has SVEP. There exist operator such that T such that $\sigma(T)$ is "somewhere" dense and have the SVEP. The unilateral shift operator on $\ell^2(\mathbb{N})$ has this property. Theorem 3.11 also holds if $\sigma(T)$ is at most countable.

Theorem 3.12: If $T \in B(H)$ is surjective and has SVEP, then it is Fredholm.
Proof. By Corollary 2.9, T is invertible and hence Fredholm. This follows from the fact that $\alpha(T) = \beta(T) = 0$ and $Ran(T) = H$ is closed.

The converse of Theorem 3.12 is not true in general. The unilateral shift T on $\ell^2(\mathbb{N})$ is Fredholm, with SVEP but is not surjective. Here, $Ran(T) = (0, x_1, x_2, x_3, \cdots)$ is closed and $\alpha(T) = \beta(T) = 1$.

Theorem 3.13: (Fredholm Alternative) A Fredholm operator $T \in B(H)$ with
(i). finite ascent satisfies $ind(T) \le 0$.
(ii). finite descent satisfies $ind(T) \ge 0$.
(iii). finite ascent and finite descent satisfies $ind(T) = 0$ and T is injective if and only if T is surjective.

We note that in finite dimensions, the Fredholm Alternative boils down to the statement that either $Tx = b$ has a unique solution or $Tx = 0$ has nontrivial solutions. The Fredholm Alternative ensures that $\sigma_e(T) = \{0\}$ for every compact operator $T \in B(H)$.

Corollary 3.14: If $T \in B(H)$ is Fredholm and $0 \in iso\ \sigma(T)$, then $asc(T) < \infty$.

Corollary 3.15: If $T \in B(H)$ is Fredholm and $0 \in iso\ \sigma(T)$, then T has SVEP at 0.

Proof: The proof follows from Corollary 3.14.

Corollary 3.16: If $T \in B(H)$ is Fredholm and $0 \in iso\ \sigma(T)$, then T is Browder.

Proof: Using the fact that T is Fredholm if and only if T^* is Fredholm and Corollary 3.14 and Corollary 3.15, we have that $asc(T) < \infty$ and $dsc(T) < \infty$. Thus T is Browder.

Remark: If $T \in B(H)$ is Fredholm, then $iso\ \sigma(T) = \pi_0(T)$.
Using this fact, we now can characterize the class of Browder operators as

$$B_b(H) = \{T \text{ Fredholm}: 0 \in \rho(T) \bigcup iso\ \sigma(T)\} = \{T \text{ Fredholm}: 0 \in \rho(T) \bigcup \pi_0(T)\}.$$

That is, T is Browder if and only if T is Fredholm and either $0 \in \rho(T)$ or 0 is an isolated point of $\sigma(T)$. From this characterization, we see that every Browder has the SVEP at 0.

4. Spectral Relations and the SVEP

We now give some relationship between some special subsets of $\sigma(T)$. It is well known that ([12], Corollary 9.2) that

$$\sigma_b(T) = \sigma_e(T) \bigcup acc\ \sigma(T).$$

We use this fact to prove the following result.

Theorem 4.1: Let $T \in B(H)$. Then $P_{00}(T) = iso\ \sigma(T) \setminus \sigma_e(T)$.

Proof: Clearly,

$$\begin{aligned} P_{00}(T) = \sigma(T) \setminus \sigma_b(T) &= \sigma(T) \bigcap (\sigma_b(T))^c = \sigma(T) \bigcap (\sigma_e(T))^c \bigcap (acc\ \sigma(T))^c \\ &= (\sigma(T) \setminus acc\ \sigma(T)) \setminus \sigma_e(T) \\ &= iso\ \sigma(T) \setminus \sigma_e(T). \end{aligned}$$

Theorem 4.1 says that $P_{00}(T)$ consists of all isolated points of finite multiplicity. If in addition T is polaroid then

$$P_{00}(T) = iso\ \sigma(T) \setminus \sigma_e(T) \subseteq \sigma_p(T) \setminus \sigma_e(T) = \pi_{00}(T).$$

This says that for a polaroid operator T the set $P_{00}(T)$ consists of all the isolated eigenvalues of finite multiplicity.

Theorem 4.2: Let $T \in B(H)$. Then
(i). $\pi_{00}(T) = \sigma_p(T) \setminus \sigma_e(T)$.
(ii). $\pi_0(T) = \pi_{00}(T) \setminus \sigma_w(T)$.

Proof: (i). Suppose that $\lambda \in \pi_{00}(T)$. Then λ is an isolated eigenvalue of T of finite multiplicity, and hence by definition λ cannot belong to $\sigma_e(T)$. This proves that $\pi_{00}(T) \subseteq \sigma_p(T) \setminus \sigma_e(T)$. On the other hand, if $\lambda \in \sigma_p(T) \setminus \sigma_e(T) \subseteq iso\ \sigma(T) \setminus \sigma_e(T)$, then by definition λ is an isolated eigenvalue of T of finite multiplicity and hence $\lambda \in \pi_{00}(T)$.

(ii). Clearly, $\pi_0(T) = iso\ \sigma(T) \setminus \sigma_w(T) = iso\ \sigma(T) \setminus \sigma_e(T) = \pi_{00}(T) \setminus \sigma_w(T)$.

In finite dimensions every eigenvalue of T is a pole of the resolvent of T.

Theorem 4.3: Let $T \in B(H)$. If T has SVEP then $P_{00}(T) = \sigma(T) \setminus \sigma_w(T)$.

Proof: From the definition for any $T \in B(H)$ we have $P_{00}(T) = \sigma(T) \setminus \sigma_b(T)$. If in addition T has SVEP, then it satisfies Browder's theorem, and hence $\sigma_b(T) = \sigma_w(T)$. This proves the claim.

5. Abstract Shift Condition, Metric Equivalence, Similarity, Quasi-similarity, Spectral Invariance and SVEP at Zero

Duggal and Kim[4] have described the class of operators $T \in B(H)$ such that $\bigcap_{n=1}^{\infty} T^n(H) = \{0\}$. Such operators are said to have an abstract shift condition (ASC). This class contains the subclass of weighted right shift operators on $\ell^2(\mathbb{N})$. Clearly operators $T \in B(H)$ in ASC have SVEP, and are not surjective. This implies that $0 \in \sigma(T)$ and $\alpha(\lambda I - T) = 0$ for every $\lambda \in \sigma(T)$. If $T \in B(H)$ is ASC then $\sigma(T)$ is connected, so that $iso\ \sigma(T) = \varnothing$ or $\sigma(T) = \{0\}$. In the second case, T is quasinilpotent and $\sigma(T) = \sigma_w(T) = \sigma_b(T)$. (see [1]).

Two operators $A, B \in B(H)$ are said to be *metrically equivalent* if $A^*A = B^*B$. This is equivalent to saying that $\|Ax\| = \|Bx\|$ for all $x \in H$. Metric equivalence of operators was introduced by Nzimbi etal [15]. Recall that an operator $T \in B(H)$ is bounded below if there exists a constant $\alpha > 0$ such that $\|Tx\| \geq \alpha \|x\|$ for all $x \in H$. This is equivalent to saying that $T^*T \geq \alpha I$. Clearly $T \in B(H)$ is bounded below if it is injective and has closed range. It is shown in ([15], Lemma 2.7) that metric equivalence preserves the property of being bounded below. Clearly if T is bounded below, then it has SVEP at 0. Metric equivalence need not preserve the spectra of operators and hence need not preserve isolated points in the spectrum. This means that metric equivalence need not preserve SVEP of operators. This means that metric equivalence need not preserve Weyl's theorem and Browder's theorem since it need not preserve ascent and descent. The identity operator I and the unilateral shift S on $\ell^2(\mathbb{N})$ are metrically equivalent. However, $asc(S) < \infty$ and $dsc(S) = \infty$, while $asc(I) = dsc(I) = 0$.

Theorem 5.1: ([15], Lemma 2.7) Let $A, B \in B(H)$ be metrically equivalent. If A is bounded below then B is bounded below.

Theorem 5.2: If $T \in B(H)$ is metrically equivalent to a unitary operator then T has SVEP at 0.
Proof. The proof follows from the fact that metric equivalence to a unitary operator is equivalent to unitary equivalence to an isometry. Since every isometry has SVEP at 0, so does T.

We note that metric equivalence preserves injectivity and hence nullity of operators in Hilbert spaces, but need not preserve co-nullity, range and rank. This implies that metric equivalence need not preserve index of injective operators.

Theorem 5.3: Let $T \in B(H)$ be a Fredholm operator. The following assertions are equivalent.
(i). T does not have SVEP at 0.
(ii). $asc(T) = \infty$.
(iii). $0 \in acc\ \sigma(T)$.

Theorem 5.4: Let $T \in B(H)$ be a Fredholm operator with $ind(T) > 0$. Then T does not have SVEP at 0.

Proof: If $ind(T) > 0$ then $asc(T) = \infty$. The claim follows from Theorem 5.3.

The condition $ind(T) > 0$ is a necessary but not a sufficient condition for T not to have the SVEP at 0. There exist operators T such that $ind(T) < 0$ and does not have the SVEP at 0. The backward shift operator $\ell^2(\mathbb{N})$ is an example.
We also note that if T and T^* have SVEP at 0, then T is Fredholm with $ind(T) = 0$.

Question 1: Does similarity or quasisimilarity preserve isolated points, Weyl's and Browder's spectra and theorems?

It is known that similarity preserves compactness and the spectral picture (that is, the spectrum, essential spectrum, nullity, co-nullity and index of operators). Similarity also preserves resolvent set, the point spectrum, continuous spectrum and the residual spectrum.

Lemma 5.5: Let $A, B \in B(H)$ be similar. Then $\pi_0(A) = \pi_0(B)$, $\pi_{00}(A) = \pi_{00}(B)$, $acc\ \sigma(A) = acc\ \sigma(B)$, $iso\ \sigma(A) = iso\ \sigma(B)$, $\sigma(A) = \sigma(B)$ and $\sigma_*(A) = \sigma_*(B)$, where * is either p, r, c or e, w, b.

Proposition 5.6: Let $A, B \in B(H)$ be similar and suppose Weyl's theorem holds for A. Then Weyl's theorem holds for B and $\pi_{00}(A) = \pi_{00}(B)$.

Proof: Using Lemma 5.5, we have $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A) = \pi_{00}(B) = \sigma(B) \setminus \sigma_w(B)$. If Weyl's theorem holds for A, then by Theorem 3.5 A has the SVEP and $\pi_0(A) = \pi_{00}(A)$. Similarity with B then implies that B has the SVEP and by Lemma 5.5 $\pi_0(A) = \pi_0(B)$ and $\pi_{00}(A) = \pi_{00}(B)$. By Theorem 3.5 we conclude that Weyl's theorem holds for B.

Corollary 5.7: Let $A, B \in B(H)$ be similar and suppose Browder's theorem holds for A. Then Browder's theorem holds for B and $\pi_0(A) = \pi_0(B)$.

Proof: Follows from Lemma 5.5 and Proposition 5.6.

Remark: It is known that quasisimilarity need not preserve spectra and parts of spectra of operators. It is also known that quasi-similarity need not preserve compactness and many other properties associated with compactness. However, quasi-similarity preserves the index of operators. It is known that SVEP is invariant under unitary equivalence, similarity and quasi-similarity. Proposition 5.6 and Corollary 5.7 need not hold in general if we replace similarity with quasi-similarity. For these results to hold, we need to assume that B has the Polaroid condition: $\pi_0(B) = \pi_{00}(B)$.

Question 2: If $A, B \in B(H)$ are similar or quasi-similar, are there isolated eigenvalues of B that are not isolated in the spectrum of A?

The following two results answer Question 2.

Theorem 5.8: Let $A, B \in B(H)$ be quasi-similar. If $iso\ \sigma(A) = iso\ \sigma(B)$, then $\pi_0(A) = \pi_0(B)$ and $\pi_{00}(A) = \pi_{00}(B)$.

Lemma 5.9: Let $A, B \in B(H)$ be quasi-similar and suppose that $\pi_0(B) = \pi_{00}(B)$. If Weyl's theorem holds for A then Weyl's theorem holds for B.

Proof: If Weyl's theorem holds for A, then by Theorem 3.5, A has the SVEP and $\pi_0(A) = \pi_{00}(A)$. Quasi-similarity with B and the polaroid condition then implies that B has the SVEP and $\pi_0(B) = \pi_{00}(B)$. This proves the claim.

Recently, Yoo[20] has proved that quasi-similarity preserves the SVEP.

Theorem 5.10: Let $A, B \in B(H)$ be Fredholm operators. If A is quasi-similar to B then $\sigma_w(A) = \sigma_w(B)$.
Proof. Suppose $\lambda \notin \sigma_w(A)$. Then $\lambda I - A$ is Weyl and quasisimilar to $\lambda I - B$. Thus $ind(\lambda I - A) = ind(\lambda I - B) = 0$. Therefore $\lambda \notin \sigma_w(B)$. This means that $\sigma_w(B) \subseteq \sigma_w(A)$. By symmetry, we also have $\sigma_w(A) \subseteq \sigma_w(B)$. This proves the claim.

Proposition 5.11: Let $T \in B(H)$. If $\sigma(T)$ is real, then T is similar to a self-adjoint operator.

Corollary 5.12: Let $T \in B(H)$. If $\sigma(T)$ is real, then T has the SVEP.
Proof. From Proposition 5.11, T is similar to a self-adjoint operator. The result then follows from the fact that every self-adjoint operator has the SVEP.

6. Discussion

An operator A may or may not have the SVEP. An operator which does not have the SVEP hides in its spectrum a pathology which does not permit the construction of a satisfactory spectral theory (see [1]). One application of the SVEP and Fredholm theory is in establishing existence and uniqueness of solutions to ordinary differential equations. For $t \in \mathbb{R}$ define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k .$$

It is well known that $x(t) = e^{tA} x_0$ is the unique solution of the linear system of the ordinary differential equations

$$\frac{dx}{dt} = Ax$$

That satisfies the initial condition $x(0) = x_0$. Many problems in the sciences can be reduced to solving an equation $Au = f$ for some bounded linear operator A and some function (or "data") f. In order for this equation to be a reasonable model of a physical situation it should have certain properties. Some of these properties can be characterized in terms of the SVEP.

7. Conclusion

In this paper, several results have been proved. It is shown that if a Fredholm operator has no SVEP at zero, then zero is an accumulation point of the spectrum of the operator. It is also shown that quasisimilar Fredholm operators have equal Weyl spectrum. A new result shows that Browder's theorem holds for T if and only if $acc\ \sigma(T) \subseteq \sigma_w(T)$. It is proved that an operator T has SVEP on $\sigma_r(T)$ and also on $\sigma_c(T)$. We have managed to show that for any isoloid operator $T \in B(H)$, that $\sigma_r(T) \bigcup \sigma_c(T) \subseteq \text{acc}\ \sigma(T)$. It is also shown that the point zero is a pole of the resolvent of T if and only if T has the SVEP at zero and $dsc(T) < \infty$.

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