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Relations between the degrees of transitive constituents of G_1 and the absolutely irreducible constituents of permutation representation G* of the group G

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Abstract
Let Ω be a finite set of arbitrary elements, G be permutation group on $\Omega, \Delta \subseteq \Omega$, $G_1 = \{1\} = \Delta_1, \Delta_2, ..., \Delta_n$ are n orbits of G, $n_i = |\Delta_i|, f_i$ is the degree of different irreducible representation of $D_1 \ldots D_r$ appearing in G*, V=V(G) be the ring of all the matrices of G, dimV=k. And also if the irreducible constituents of permutation representation G* are all different, then the expression $q = n^{k-2}\prod_{i=1}^{k} n_i / \prod_{i=1}^{k} f_j^{e_j^2}$ is a rational integer.

Keywords: irreducible constituents orbits, transitive constituents, permutation representation degree.
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1. Introduction
In 1943. R. Brauer studied about permutation groups and finid the permutation groups of prime degree and related classes of Groups (See) [4]. In 1928, M.J. Weiss studied the primitive groups which contain substitutions of prime order p and of degree 6p or 7p, and in 1934 he worked on simply transitive groups and obtained the beautiful consequences, (See) [25, 26]. In 1906, W. Burnside introduced d researched the a bout transitive groups, of prime degree (See) [2], and in 1921, he worked about the certain simply- transitive permutation group and obtained a beautiful consequences (See) [3].
From years 1906 to 1936. W.A. Manning studied about primitive groups and finding the primitive groups of classes six, ten, twelve and fifteen. (See) [11-18]. In 1937 J.S. Frame determined the degrees of the irreducible components of simply transitive permutation groups, and in 1941, he obtained the double cosets of a finite groups (See) [5, 6]. And also in 1952, he finding the irreducible representation extracted from two permutation groups (See [7]). G.A. Miller (1897 & 1915), (See) [19, 20], E.T. Parker (1954), (See) [21], M. Suzuki (1962), (See) [23], J.G. Thompson (1959), (See) [24], M.J. Weiss (1928), (See) [25, 26], H. Wielandt (1935 & 1956) (See) [27, 28] and H. Zassenhaus (1935), (See) [30] are studied about transitive and primitive groups and their obtaind the beautiful and more consequence. In 1935, N.Itô, studied on primitive permutation groups and in 1962 he obtained transitive simple permeation groups of degree 2p, (See) [9, 10].
Now in this paper we will study the relations between the degrees n_i of the transitive constituents of G_1 and the degrees f_i of the absolutely irreducible constituents D_i of the permutation representation G* of the group G, and the end we expressed this relations in expression $q = n^{k-2}\prod_{i=1}^{k} n_i / \prod_{i=1}^{k} f_j^{e_j^2}$, and also our prove that the expression q is a rational integer.

2. Periliminares: In this chapter we study the elementary properties, lemmas and theorems, whose we will used the next chapter.

2.1. Elementary notions and definitions: Let Ω be a finite set of arbitrary elements which for natural numbers 1,2,…,n as the points and Δ subset of Ω. Then a permutation on Ω is a one-to-one mapping of Ω onto itself. We denote the image of the point $\alpha \in \Omega$ under the permutation p by α^{p}.We write $P=\begin{pmatrix}1 & 2 & \dots & n\\ 1^{p} & 2^{p} & \dots & n^{p}\end{pmatrix}=\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}p$.. We define the product pq of two permutations p and q on Ω by the formula $\alpha^{pq}=(\alpha^{p})^{q}$. trivialy pq is again a permutation on Ω.
With respect to the operation above, all the permutations on Ω form a group, the symmetric group S^{G}. Let G be a permutation group on Ω, in short $G \le S^{\Omega}$. We say that a set $\Delta \subseteq \Omega$ is a fixed block of G or is fixed by G if $\Delta=\Delta^{G}$. Then each $g \in G$ induces a permutation on Δ which we denote by g^{Δ}. We call the totality of g^{Δ}'s formed for all $g \in G$ the constituent G^{Δ} of G on Δ(for example $G=G^{\Delta}$). G^{Δ} is a permutation group on Δ.Obviously the mapping $g \to g^{\Delta}$ is a homomorphism $:G \xrightarrow{\sim} G^{\Delta}$. If this mapping is an isomorphism, that is, $|G^{\Delta}|=|G|$, then the constituents G^{Δ} is called faithful. Every group G on Ω has the trivial fixed blocks ϕ on Ω. If it has no others it is called transitive. Otherwise it is called intransitive. Accordingly, a constituent G^{Δ} is transitive precisely when Δ is a minimal fixed block ($\Delta \ne \phi$). In this case Δ is called an orbit or set of transitivity of G. Every permutation g on Ω can be regarded in the following way as a liner substitution in $|\Omega|$=n variables. the variables $X_1,\dots,X_n$ are taken as points. We form colum vectors $\cancel{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $\cancel{x}^{g}=\begin{bmatrix}x_1^{g}\\ \vdots\\ x_n^{g}\end{bmatrix}=g^{*}_{\cancel{x}}$ where $g^{*}=\left(\delta_{\alpha^{g},\beta}\right)_{\alpha,\beta=1,\dots,n}$ is the n by n matrix corresponding to the linear transformation $\cancel{x} \to \cancel{x}^{g}$. ($\delta_{\alpha\beta}=\begin{cases}1 & , \ \alpha=\beta\\ \circ & , \ \alpha \ne \beta\end{cases}$ is the well known Kronecker symbol). We call g* the permutation matrix corresponding to g. such a matrix contains exactly one 1 in each row and column and zeros every where else. In addition, every permutation matrix g* is orthogonal, i.e., the transpose $g^{*\prime}$ of g* is identical with its inverse: $g^{*\prime}=g^{*-1}$, we obtain a faithful representationof S^{G} by $g \to g^{*}$. Now let $G \le S^{G}$. By G* we denote the group of all matrices g* with $g \in G$. Obviously G* is isomorphic to G. We call G* the permutation representation of G.

2.2. Theorem (See [22]). If a transitive permutation group G is regarded as a matrix group G*, then the matrices which commute with all the matrices of G* form a ring V=V(G). We call V "the centralizer ring corresponding to G". V is a vector space over the complex number field which has the matrices $B(\Delta)$ corresponding to the orbits Δ of G_1 as a linear basis. In particular, the dimension of V coincides with the number k of orbits of G_1.

Proof: See [29], Theorem 28.4]
Let $D_1,\dots,D_r$ be the different irreducible representations appearing in G* where D_1 is the identity representation. In the following we always denote by f_i the degree of D_i (i=1,…,r), and by e_i the multiplicity of D_i in G*. In particular, we have $e_1=f_1=1$ and $\sum_i e_i f_i = n$, the reduction of G* gives for an appropriately chosen unitary n by n matrix U :

$$U^{-1}G^{*}U=[D_1,\underbrace{D_2,\dots,D_2}_{e_2},\dots,\underbrace{D_n,\dots,D_n}_{e_r}].$$

2.3. Proposition: Dim (V) = $\sum_{i=1}^{r} e_i^2 = k$.

Proof: See ([29], Proposition 29.2)

2.4. Definition: Let G is transitive and consider the orbits of G_1. With each of these orbits Δ (including the trivial $\Delta=\{1\}$) we associate in the following way a matrix $B(\Delta)=\left(V_{\alpha,\beta}^{\Delta}\right)$, $\alpha,\beta=1,\dots,n$, with elements:

$$V_{\alpha,\beta}^{\Delta}=\begin{cases}1, & \text{if there exists } g \in G \text{ and } \delta \in \Delta \text{ with } 1^{g}=\beta \text{ and } \delta^{g}=\alpha.\\ 0, & \text{otherwise}\end{cases}$$

Thus, in the first colum of $B(\Delta)$ we have exactly those $V_{\alpha,\beta}^{\Delta}=1$ for which $\alpha \in \Delta$ holds. If $\Gamma \ne \Delta$, the ones of $B(\Gamma)$ and $B(\Delta)$ do not occur in the same place. On the other hand, for each place (α,β) there is an orbit Δ of G, (namely, the one in which the $\alpha^{g^{-1}}$ with $1^{g}=\beta$ lies) such that $B(\Delta)$ has 1 in this position.

2.5. Proposition: Let M be the n by n matrix whose elements are all 1.then $\sum_{\Delta} B(\Delta) = M$. (Here the summation is over all orbits of G_i)

Proof: See ([29], proposition 28.2).

2.6. Theorem. V is commutative if and only if all the e_i=1.

Proof: See ([29], Theorem 29.3)

2.7.Theorem. V is commutative if and only if the class matrices $E_i = \sum_{g \in C_i} g^*$ (i = 1,..,n) whose C_i be the ith class of conjugate elements of G, generate V, i.e., when each $B \in V$ has a (not necessarily unique) representation $B = \sum_{i=1}^{r} z_i E_i$.

Proof: See ([29], Theorem 29.8)

2.8. Theorem: Let Γ and Δ be two orbits of G_1. then $Tr(B(\Gamma)'B(\Delta) = \begin{cases} \circ & , \quad \Gamma \neq \Delta \\ |\Gamma| n & , \quad \Gamma = \Delta \end{cases}$

Proof: See ([29], Theprem 28.10)

3. Main result

In this section by used of the theorems and lemmas of section 2, we prove that the following main theorem:

3.1.Main theorem: Let Ω be a finite set of arbitrary elements, G be permutation group on $\Omega, \Delta \subseteq \Omega$, $G_1 = \{1\} = \Delta_1, \Delta_2, ..., \Delta_n$ are n orbits of G, $n_i = |\Delta_i|, f_i$ is the degree of different irreducible representation of $D_1 \ldots D_r$ appearing in G*, V=V(G) be the ring of all the matrices of G, dimV=k. And also if the irreducible constituents of permutation representation G* are all different, then the expression $q = n^{k-2} \prod_{i=1}^{k} n_i / \prod_{i=1}^{k} f_j^{e_j^2}$ is a rational integer.

Before from proof of the main theorem, we first the prove the following theorems. And the end we consequence the main theorem.

3.2. Theorem: (A) If the irreducible constituents of G* are all different, i.e., if all the multiplicities e_i=1, then the rational number $q = n^{k-2} \prod_{i=1}^{k} \frac{n_i}{f_i}$ is an integer.

(B) If in addition the k numbers n_i are all different, then q is a square.

(C) If the irreducible constituents of G* all have rational characters, then q is a square. The hypothesis is always fulfilled if the degrees f_i are all different.

Notice: It is not known whether (B) is true for k>3.

Proof of 3.2.(A). It suffices to show that q is an algebraic integer. The notation of the preceding section is continued.

Let U again be the unitary transformation matrix introduced in §2. From the hypothesis e_i=1 it follows that every matrix $M=U^{-1}BU$ with $B \in V$ has diagonal form. Because $B_i = B(\Delta_i) \in V(G)$ we have in particular

$M_i = U^{-1}BU = [w_{1i}, w_{2i}\Im_{f_2}, ..., w_{ki}\Im_{f_k}]$, which $\Im_{f_i}$ is the f_i by f_i identity matrix.

Let w_i be the diagonal elements of the matrix $M = U^{-1}BU$, for arbitrary $B \in V$. We put

$$B = \sum_i z_i B_i, \qquad z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix},$$

$N = [1, n_2, ..., n_k], \quad F = [1, f_2, ..., f_k]$, and $I=(w_{ij})$; i,j=1,…,k. From $M = U^{-1}BU$ it follows that $\overline{M}'M = U^{-1}\overline{B}'BU$, since U was assumed unitary. With the aid of 2.8 we now obtain

$$\bar{z}'Nnz = \sum_{i=1}^{k} \bar{z}_i z_i n_i n = \sum_{i,j} \bar{z}_i z_j Tr(B_i' B_j) = Tr(\overline{B}'B) = Tr(\overline{M}'M) = \sum_i \overline{w}_i w_i f_i = \overline{w}'Fw.$$

Because $w_i = \sum_j z_j w_{ij}$, i.e., $w = Iz$, we therefore have $Nn = \bar{I}'FI$. By taking the determinant we get $n^k \prod_i n_i = |Nn| = |F|\,|\bar{I}'I| = \prod_i f_i\,|\bar{I}'|\,|I|$.

The w_{ij}, as eigenvalues of the matrix B_j which has integer coefficients, are algebraic integers, and therefore | I | and $|\bar{I}|$ are also algebraic integers.

We wish to show that $|I|$ is divisible by n. by 2.5, $\Sigma_j B_j$ =M where M is the n by n matrix consisting of n^2 ones. M has the eigenvalue n occurring with multiplicity 1 belonging to the eigenvector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. Its remaining eigenvalues are 0. In the diagonal matrix $\Sigma_j M_j$, n therefore appears exactly once, the remaining elements being 0. Therefore $\Sigma_j w_{ij} = n$ for i=1 and =0 for the remaining. This implies

$$|I| = \begin{vmatrix} n & w_{12} & \cdots & w_{1k} \\ 0 & w_{22} & \cdots & w_{2k} \\ \vdots & & & \\ 0 & w_{k2} & \cdots & w_{kk} \end{vmatrix} \equiv 0 \quad (\text{mod } n).$$

Hence $q = n^{-2}|\bar{I}||I|$ is an algebraic integer, hence also a rational integer.

Proof of 3.2(C). (a) Because of the hypothesis $e_1 = \cdots = e_k = 1$, the commutativity of V follows by 2.4. Theorem 2.7 yields the existence of k class matrices $C_1, \ldots, C_k$ and of complex numbers x_{ij} such that $B_i = \sum_{j=1}^{k} x_{ij} C_j \qquad (i = 1, \ldots, k)$. Conversely by 2.2 there are also x'_{ij} with $C_i = \sum_{j=1}^{k} x'_{ij} B_j$. The x_{ij} are, by well-known theorems of linear algebra, uniquely determined and rational, since the matrices B_i and C_j are rational.

(b) By hypothesis all the irreducible characters appearing in G* are rational. Thus the matrics $U^{-1}C_j u$ appearing in the proof of 2.6 are also rational. By (a) the matrices $M_i = U^{-1}B_i U = \sum_i x_{ij} U^{-1} C_j U$ are then rational. The w_{ij} are therefore rational. Since the w_{ij} were already in the proof of 3.2 (A) shown to be algebraic integers, they are rational integers. $|I| = |w_{ij}|$ is therefore a rational integer. Since n divides $|I|$, $n^{-1}|I|$ is also a rational integer, and q is therefore a square as was asserted.

(c) The hypothesis that all the irreducible characters appearing in G^* are rational is fulfilled if the degrees f_i of the irreducible constituents of G^* are all different. For since G* is rational, with each irreducible representation D_i all representations conjugate to it appear in G^*. Because all of the f_i are different, these coincide with D_i, and X_i is therefore rational.

We assume in the sequel that V is commutative, i.e., that the irreducible constituents of G^* are all different, and draw some conclusions from 3.2.

Theorem 3.3. If the nontrivial irreducible representations appearing in G^* all have the same degree f, then the orbits of G_1 different from 1 all have the same length, which also equals f. Hence G is either regular and Abelian (f = 1) or (3/2)-fold transitive (f >1).

Proof. From $f_i = f$ for $i = 2, \ldots, k$ and 3.2(A) it follows that f^{k-1} divides $n^{k-2}\prod_i n_i$. Since f divides n-1, f and n are relatively prime, hence f^{k-1} divides $\prod_i n_i$. In addition we have

$$(k-1)f = n-1 = \sum_{i=2}^{k} n_i, \qquad \text{i.e.,} \qquad \frac{1}{k-1}\sum_{i=2}^{k} n_i = f.$$

The geometric mean of the n_i is at most equal to the arithmetic mean, thus $\prod_i n_i \le f^{k-1}$. With the help of the divisibility property it now follows that $\prod_i n_i = f^{k-1}$. The geometric and arithmetic mean of the n_i therefore coincide. This occurs, however, only if $n_i = f$, as was to be proved.

From theorem 3.3 it follows immediately that:

Theorem 3.4. If k=3 and $n_2 \neq n_3$, then also f_2 / f_3. Hence 3.1 (B) holds for k=3.

It is not known if in general in general when the n_i are all different (i=2, …,k) it follows that the f_i are also all different. Should this prove correct, the conjecture 3.2(B) would be true.

In the proof of theorem 3.2 we used the matrix equation $Nn = \bar{I}'FI$ only to compare the determinants. More precise information can be obtained from the elementary divisors. Since the k-th elementary divisor of a product of matrices is known to be divisible by the k-th elementary divisor of each factor, the k-th elementary divisor of Nn is divisible by the k-th elementary divisor of F. In other words:

Theorem 3.5. If, under the assumptions of 3.2(A), there are l of the numbers f_i divisible by a given prime power p^a then there are at least l of the numbers nn_i divisible by p^a.

Now by used from theorems 3.2, 3.3, 3.4 and 3.5 and as generalization of theorem 3.2to the case where some multiplicities $e_i > 1$, we follows that :

Main theorem 3.6. the expression $q = n^{k-2} \prod_{i=1}^{k} n_i / \prod_{i=1}^{k} f_j^{e_j^2}$ is a rational integer.

If in addition the irreducible constituents of G^* have rational characters, then q is square. The hypothesis is always satisfied if the degrees of the different irreducible constituents of G^* are all different.

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