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Some Recursive relations of Chebyshev polynomials using standard Recurrence formulas

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Abstract
Chebyshev polynomials make a sequence of orthogonal polynomials, which has a big contribution in the theory of approximation. In this paper, after providing brief introduction of Chebyshev polynomials, we have used two Recursive relation of Chebyshev polynomials in finding some more similar relations.

Keywords: Chebyshev polynomials, orthogonal, approximation, Recursive relation, differential equations

1. Introduction
Multiple angle cosine formulas are very common in the study of mathematics. Using single and double angle formula, we can go through the higher angles formulas as well. For example let's have a look of certain formulas-

$$\cos 0y = 1$$
$$\cos 1y = \cos y$$
$$\cos 2y = 2\cos^2 y - 1$$
$$\cos 3y = 4\cos^3 y - 3\cos y$$
$$\cos 4y = 8\cos^4 y - 8\cos^2 y + 1$$
$$\cos 5y = 16\cos^5 y - 20\cos^3 y + 5\cos y$$
$$\cos 6y = 32\cos^6 y - 48\cos^4 y + 18\cos^2 y - 1$$

Now let us replace ($\cos y$) as x and say $T_m(\cos y) = \cos my$ then above equations becomes as-

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

These polynomials are called as Chebyshev polynomials of first kind. For the sake of convenience we call them Chebyshev polynomial.

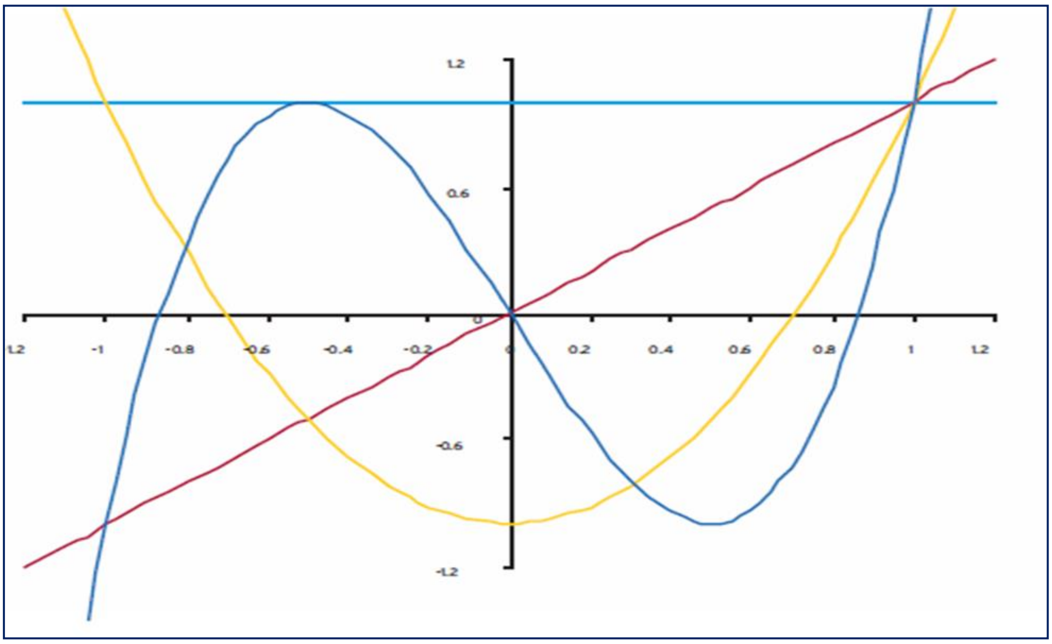

Fig 1: First Four Chebyshev polynomials

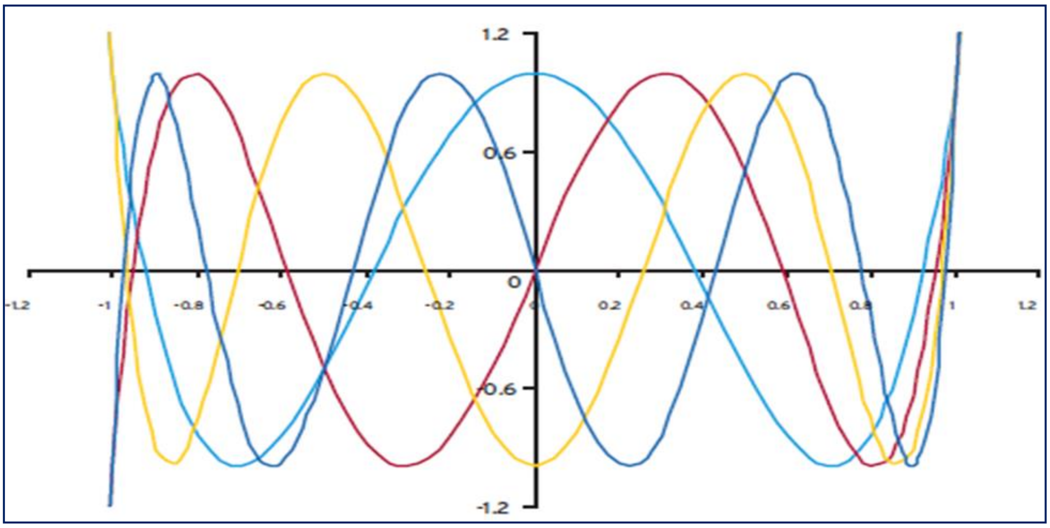

Fig 2: Second Four Chebyshev polynomials

2. The Chebyshev differential equation:
The Chebyshev differential equation is-

$$(1-x^2)y''-xy'+m^2y=0$$

Where m is any whole number.
Let x = cos t and we get-

$$y''+m^2y=0$$

Its general solution is-

$$y=a\cos(mt)+b\sin(mt)$$

Writing in the form of x we get-

$$y=a\cos(m\cos^{-1}x)+b\sin(m\cos^{-1}x)$$

$$y=aT_m(x)+bU_m(x)$$

Here $|x|\le 1$

Where $T_m(x)$ and $U_m(x)$ are defined as Chebyshev polynomials of the first and second kind of degree n, respectively.

3. Rodrigue's formula
The Chebyshev polynomials $T_m(x)$ can be obtained by means of Rodrigue's formula-

$$T_m(x)=\frac{(-2)^m m!}{(2m)!}\sqrt{1-x^2}\frac{d^m}{dx^m}(1-x^2)^{(m-1/2)}$$

Where m is a whole number.

4. Recurrence Relations

The first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials $T_m(x)$, $m \geq 2$ can be obtained by means of the recurrence formula-

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x) \quad (1)$$

The derivative of $T_m(x)$ with respect to x can be obtained from-

$$(1 - x^2)T'_m(x) = -mxT_m(x) + mT_{m-1}(x) \quad (2)$$

Using (1) we get,

$$T'_{m+1}(x) = 2T_m(x) + 2xT'_m(x) - T'_{m-1}(x) \quad (a)$$

$$xT'_{m+1}(x) = 2xT_m(x) + 2x^2T'_m(x) - xT'_{m-1}(x)$$

$$xT'_{m+1}(x) = T_{m+1}(x) + T_{m-1}(x) + 2x^2T'_m(x) - xT'_{m-1}(x)$$

$$x[T'_{m+1}(x) + T'_{m-1}(x)] - [T_{m+1}(x) + T_{m-1}(x)] = 2x^2T'_m(x) \quad (b)$$

$$x[T'_{m+1}(x) + T'_{m-1}(x)] - [T_{m+1}(x) + T_{m-1}(x)] = 2mxT_m(x) - mT_{m-1}(x) - 2T'_m(x) \quad (c)$$

From equation (2) we have-

$$2(1 - x^2)T'_m(x) = -m2xT_m(x) + 2mT_{m-1}(x)$$

$$2(1 - x^2)T'_m(x) = -m[T_{m+1}(x) + T_{m-1}(x)] + 2mT_{m-1}(x)$$

$$2(x^2 - 1)T'_m(x) = m[T_{m+1}(x) - T_{m-1}(x)] \quad (d)$$

From (1)-

$$T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x)$$

$$T_{m+1}(x) = xT_m(x) - \sqrt{(1 - x^2)[1 - \{T_m(x)\}^2]} \quad (e)$$

From equation 2-

$$(1 - x^2)T'_m(x) = -mxT_m(x) + mT_{m-1}(x)$$

$$(x^2 - 1)T''_m(x) + 2xT'_m(x) = mxT'_m(x) + mT_m(x) - mT'_{m-1}(x)$$

$$(x^2 - 1)T''_m(x) + mT'_{m-1}(x) = (m - 2)xT'_m(x) + mT_m(x) \quad (f)$$

5. Conclusions

Here we derived some more recurrence relations using two fundamental recursive Formulas. These relations are useful in finding higher order Chebyshev polynomials or its derivative as well. We can use the above relations whenever needed some Chebyshev polynomial or its derivative with corresponding provided related polynomials. These Chebyshev polynomials provide a min/max implementation to many numerical solutions.

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