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Product of finitely permutable groups

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Abstract
In this paper we show that if A1,A2,…,An are finitely many pairwise permutable abelian min-by-max subgroups of the group G such that G=A1…An, then J(G)=J(A1)…J(An), which J(G) finitely residual of a group G.

Keywords: Minimal condition, maximal condition, finite residual group
2000 Mathematics subject classification: 20B32, 20D10

1. Introduction

In 1940 G. Zappa (See [21]) and in 1950 J. Szip (See [20]) studied bout products of groups concerned finite groups. In 1961 O.H. Kegel (See [8]) and in 1958 H. Wielandt (See [10]) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups.

In 1955 N. Itô (See [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (See [18]) and L. Redei (1950)(See [19]) considered products of cyclic groups, and around 1965 O.H. Kegel (See [23, 24]) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (See [16]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition. He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [17, 1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See [17])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (See [1-4, 6]), N.S. Chernikov (See [5]), S. Franciosi, F. de Giovanni (See [3, 6, 25-29]), O.H. Kegel (See [8]), J.C. Lennox (See [12]), D.J.S. Robinson (See [9, 14]), J.E. Roseblade (See [13]), Y.P. Sysak (See [30-33]), J.S. Wilson (See [34]), and D.I. Zaitsev (See [11, 15]).

Now, in this paper, we study the residual finite group and min-by-max subgroups of the group G and its relations, and the end we prove that if $A_1, A_2, ..., A_n$, are finitely many pairwise permutable abelian min-by-max subgroups of the group G such that G is the products of $A_1, ..., A_n$. Then G is soluble min-by-max-group and J(G) is products of $J(A_1), ..., J(A_n)$, i.e. $J(G) = J(A_1)...J(A_n)$. For do this, in chapter 2 we express the elementary lemmas and Theorems and in chapter three we prove the main Theorem.

2. Preliminaries: (Elementary properties and Theorems.)

In this chapter we express the elementary Lemma and difinitons whose used in prove the Man Theorem in chapter

2.1. Theorem (See [11, 12])**:** If the soluble-by-finite group G=AB is the product of two polycyclic-by-finite subgroups A and B, then G is polycyclic-by-finite.

Proof: Assume that G it not polycyclic-by-finite. Then G contains an abelian normal section U/V which is either torsion-free or periodic and is not finitely generated. Clearly the factorizer of U/V in G/V is also a counterexample. Hence we may suppose that G has a triple factorization G=AB=AK=BK, Where K is an abelian normal subgroup of G which is either torsion-free or periodic. By Lemma 1.2.6(i) of [4] (See also [17]) the group G satisfies the maximal condition on normal subgroups, so that it contains a normal subgroup M which is maximal with respect to the condition that G/M is not polycyclic-by-finite. Thus it can be assumed that every proper factor group of G is polycylic-by-finite.

2. 2. Theorem (See [15]): Let the soluble group G=AB be the product of two subgroups A and B with finite abelian section rank. If at least one of the factors A and B has an ascending normal series with central or periodic factors, then G also has finite abelian section rank.

Proof: See ([4], Theorem 4.6.10).

2. 3. Theorem (See [6]): Let the group G=AB=AK=BK be the product of three nilpotent subgroups A, B, and K, where K is normal in G. If K is minimax, then G is nilpotent.

Proof: See ([4], Theorem 6.3.4).

2. 4.Theorem (See [6])**:** Let the group G=AB=AK=BK be the product of two subgroups A and B and a minimax normal subgroup K of G.
(i) if A,B, and K are locally nilpotent, then G is locally nilpotent.
(ii) If A, B, and K are hypercentral, then G is hypercentral.

Proof: See ([4], Theorem 6.3.7).

2.5 Lemma: Let the group G=AB be the product of two abelian subgroups A and B such that $A_G=B_G=1$. Then the following hold.
(i) $A\cap B=Z(G)=1.$
(ii) $A\cap C_G(G')=B\cap C_G(G')=1,$ and in particular $A\cap G'=B\cap G'=1.$
(iii) The factorizer $X=X(G')$ of G' does not have non-trivial normal subgroups which are contained in A or B, so that in particular Z(X)=1.
(iv) The FC-centre of G is trivial.

Proof: (i) They Lemma 2.7 we have that
$$Z(G)=(A\cap Z(G))(B\cap Z(G)); A_GB_G=1.$$
Hence Z(G)=1. Moreover, $A\cap B$ in contained in Z(G) and so is also trivial.
(ii)This follows from the first part of the proof of Lemma 2.9.
(iii) Let N be a normal subgroup of X contained in A. Then G' normalizes N, so that by (ii)
$$[N,G']=N\cap G=A\cap G'=1$$
Therefore N is contained in $A\cap C_G(G')=1$

(iv) Let a be an element of $A\cap F$, where F is the FC-centre of G. Since G' is abelian by Theorem 2.5, the mapping $\varphi: x\mapsto [x,a]$ is a G epimorphism from G' onto $[G',a]$. Hence $C_{G'}(a)=\ker\varphi$ is a normal subgroup of G, and the abelian groups $G'/C_{G'}(a)$ and $[G',a]$ are G isomorphic. The factorizer X=X(G') of G' has the triple factorization
X=A*B*=A* G'=B* G',
Where $A^*=A\cap BG'$ and $B^*=B\cap AG'$. As $G'/C_{G'}(a)$ is finite, it follows from Theorem 2.13 that $X/C_{G'}(a)$ is nilpotent. Therefore $[G',a]$ is contained in some term of the upper central series of X. Since Z(X)=1 by(iii), we have $[G',a]=1$ and so a belongs to $A\cap C_G(G')$. Thus a=1 by (ii), and hence $A\cap F=1$. Similarly $B\cap F=1$. It follows from Lemma 2.8 that $F=(A\cap F)(B\cap F)=1$.

2.6 Theorem: (See [22]): Let the group G=AB$\neq 1$ be the product of two abelian subgroups A and B, at least one of which has finite section rank. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof: Assume that $A_G=B_G=1,$ so that $A\cap G'=B\cap G'=1.$ by Lemma 2.15(ii). The factorizer $X=X(G')$ has the triple factorization
$$X=(A\cap BG')(B\cap AG')=(A\cap BG')G'=(B\cap AG')G',$$
And its centre is trivial by Lemma 2.15(iii). The subgroups $A\cap BG'$ and $B\cap AG'$ are isomorphic, and hence both have finite section rank. By Theorem 2.12 the metabelian group X has finite abelian section rank, and hence is hypercentral by Theorem 2.14. In particular $Z(X)\neq 1,$ a contradiction.

2.17 Theorem: (See [35])**:** Let the group G=$A_1\ldots A_t$ be the product of finitely many pairwise permutable abelian minimax subgroups $A_1,\ldots,A_t$. Then G is a soluble minimax group.
Proof: Assume that the theorem is false, and let G= $A_1\ldots A_t$ be a counterexample for which the sum $t+\sum_{i=1}^{t}m(A_i)$ is minimal. Suppose that there are indices i<j such that $D=A_i\cap A_j$ is infinite. Then
$$D^G=D^{A_1\ldots A_t}=D^{A_1\ldots A_{i-1}A_{i+1}\ldots A_{j-1}A_{j+1}\ldots A_t}\leq A_1\ldots A_i\ldots A_{j-1}A_{j+1}\ldots A_t.$$
It follows that D^G is a soluble minimax group. On the other hand, the factor group $\overline{G}=G/D^G$ is also a soluble minimax group since $m(\overline{A}_i)<m(A_i)$. This contradiction shows that $A_i\cap A_i$ is finite if $i\neq j$.
Let J_i be the finite residual of A_i for every i=1,…,t. It follows from lemma 2.15 that J_iJ_j is the finite residual of the soluble minimax group A_iA_j, so that it is abelian. Hence $L=\langle J_1,\ldots,J_t\rangle$ is an abelian group satisfying the minimal condition. As $[A_i,J_j]\leq J_iJ_j\leq L,$ the subgroup L is normal in G. Assume that $J_i\neq 1$ for some i. Then $m(A_iL/L)<m(A_i),$ and so G/L is a soluble minimax group. This contradiction proves that $J_i=1$ for each i. In

particular the maximum periodic normal subgroup E of A_1A_2 is finite. If A_1A_2=E, then the soluble minimax group by Corollary 2.10 Thus E is properly contained in A_1A_2, and by Theorem 2.16 we may suppose that A_1E/E contains a non-trivial normal subgroup N/E of
$A_1A_2/E=(A_1E/E)(A_2E/E)$.
As A_1A_2/E has no finite-non-trivial normal subgroups, N/E must be infinite. Moreover, the index $|N : N \cap A_1| = |A_1N : A_1| \leq |A_1E : A_1|$ is finite. If M is the core of $N \cap A_1$ in A_1A_2, then N/M has finite exponent and hence is finite. Therefore M is an infinite normal subgroup of A_1A_2 contained is A_1. Since
$M^G = M^{A_3 \ldots A_t} \leq A_1A_3 \ldots A_t$, it follows that M^G is a soluble minimax group. As above, G/M^G is also a soluble minimax group since $m(A_1 M^G/M^G) < m(A1)$ This contradiction proves the theorem.

3. Main Result: In this chapter by used the Lemmas and Theorems of chaper 2, we prove the Basic theorem of this paper as follows.

3.1. Main Theorem: Let the group G= $A_1 \ldots A_t$ be the product of finitely many pairwise permutable abelian min-by-max subgroups $A_1, \ldots, A_t$. Then G is a soluble min-by-max group and $J(G)=J(A_1) \ldots J(A_t)$.

Proof: It follows from Theorem 2.17 that G is soluble minimax group, and hence J=J(G) is abelian. Put $J_i = J(A_i)$ for each i=1,…,t. Then $L = J_1 \ldots J_t$ is contained in J. Let I be the finite residual of A_iA_j. The factorizer X=X(I) of I in A_iA_j has the triple factorization

$$X = A_i^* A_j^* = A_i^* I = A_j^* I,$$

where $A_i^* = A_i \cap A_j I$ and $A_j^* = A_j \cap A_i I$. It follows that J_i and J_j are contained in Z(X), and the factor group X/ J_iJ_j is polycyclic by Theorem2.11. Therefore J_iJ_j is the finite residual of X and so $J_iJ_j = I.$ Thus $\lfloor A_i, J_j \rfloor \leq J_iJ_j \leq L,$ and hence L is normal in G. The factor group A_iL/L is polucyclic for every $i \leq t$,and hence also $G/L = (A_1L/L) \ldots (A_tL/L)$ is polycyclic by Theorem 2.10. This proves that G is a min-by-max group and $J=L=J_1 \ldots J_t$.

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