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AS Abedl-Rady
Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

SZ Rida
Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

AAM Arafa
Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt

HR Abedl-Rahim
Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

Correspondence:
AS Abedl-Rady
Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt

Approximate Analytical Solutions of Fractional Nonlinear Physical Differential Equations

AS Abedl-Rady, SZ Rida, AAM Arafa and HR Abedl-Rahim

Abstract
In this paper, we propose a new approximate method, namely homotopy perturbation natural transform method (HPNTM) to solve fractional differential equations arising in physics. This method is a combination of the natural transform method and the homotopy perturbation method. The results reveal that the method is very effective, simple and can be applied to other physical differential equations. The fractional derivatives are described in the Caputo sense.

Keywords: Natural transform, Homotopy perturbation natural transform method (HPNTM), Fractional sinh-Gordon equation, Fractional Sharma-Tasso-Olver equation

1. Introduction

The natural transform, initially was defined by Waqar *et al.* [1] as the N - transform, which studied their properties and applications. Later, Belgacem *et al.* [2, 3] defined its inverse and studied some additional fundamental properties of this integral transform and named it the natural transform. Applications of natural transform in the solution of differential and integral equations and for the distribution and Bohemians spaces can be found in [3-10]. Now, we mention the following basic definitions of natural transform and its properties are introduced as follows:

Definition 1.1 [11].

Over the set of functions

$$A=\{f(t):\exists M,\tau_1,\tau_2>0,|\ f(t)\ |<Me^{|t|/\tau_j},\ if\ t\in(-1)^j\times[0,\infty)\}$$

The natural transform of $f(t)$ is $N[f(t)]=R(s;u)=\int_0^\infty f(ut)e^{-st}dt, u>0, s>0$ (1)

Where, $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables u and s are the natural transform variables.

Theorem 1.2: We derives the relationship between Natural and Laplace, Sumudu transform in successive theorems [11] as follow:

1- If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A, $G(u)$ is Sumudu transform then,

$$N[f(t)]=R(s;u)=\frac{1}{u}\int_0^\infty f(t)e^{-\frac{st}{u}}dt=\frac{1}{u}F(\frac{s}{u}), \tag{2}$$

2. If $R(s,u)$ is natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A then, $G(u)$ is Sumudu transform of function $f(t)$ in A, then:

$$N[f(t)]=R(s;u)=\frac{1}{s}\int_0^\infty f(\frac{ut}{s})e^{-t}dt=\frac{1}{s}G(\frac{u}{s}) \tag{3}$$

3- If $f^{n}(t)$ is the *n*th derivative of function $f(t)$ then, its natural transform is given by:

$$N[f^{n}(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0), n \geq 1 \quad (4)$$

4. If $F(s,u)$, $G(s,u)$ are the natural transform of respective functions $f(t)$, $g(t)$ both defined in set A then,

$$N[f * g] = uF(s,u)G(s,u) \quad (5)$$

Where $f * g$ is convolution of two functions f and g.

5. If $N[f(t)]$ is the natural transform of the function $f(t)$, then the natural transform of fractional derivative of order α is defined as:

$$N[f^{(\alpha)}(t)] = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} f^{(k)}(0) \quad (6)$$

6. Let the function $f(t)$ belongs to set A be multiplied with weight function $e^{\pm t}$ then,

$$N[e^{\pm t} f(t)] = \frac{s}{s \mp u} R[\frac{s}{s \mp u}] \quad (7)$$

7. Let the function $f(at)$ belongs to set A, where a is non-zero constant then,

$$N[f(at)] = \frac{1}{a} R[\frac{s}{a}, u] \quad (8)$$

8. If $w^n(t)$ is given by $w^n(t) = \int_0^t \ldots \int_0^t f(t)(dt)^n dt$, then, the natural transform of $w^n(t)$ is given by:

$$N[w^n(t)] = \frac{u^n}{s^n} R(s,u) \quad (9)$$

9. The natural transform of T-periodic function $f(t) \in A$ such that $f(t+nT) = f(t), n = 0,1,2,\ldots$ is given by:

$$N[f(t)] = R(s,u) = [1 - e^{\frac{-sT}{u}}]^{-1} \frac{1}{u} \int_0^T e^{-\frac{st}{u}} f(t)dt \quad (10)$$

10. The function $f(t)$ in set A is multiplied with shift function t^n, then,

$$N[t^n f(t)] = \frac{u^n}{s^n} \frac{d}{du^n}^n u^n R(s,u) \quad (11)$$

2 Homotopy perturbation natural transform method (HPNTM)

To illustrate the basic idea of this method, we consider the following nonlinear fractional differential equation:

$$D_t^\alpha U(x,t) + L(U(x,t)) + F(U(x,t)) = q(x,t),\ t > 0, 0 < \alpha < 1 \quad (12)$$

Subject to initial condition:

$$U(x,0) = f(x)$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional Caputo derivative of the function $U(x,t)$,

L is the linear differential operator, F is the nonlinear differential operator, and $q(x,t)$ is the source term. Now, applying the natural transform on both sides of (12) we have:

$$\frac{s^\alpha}{u^\alpha} N[U] - \sum_{k=0}^{\alpha-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} U^{(k)}(0) + N[LU] + N[FU] = N[q(x,t)] \quad (13)$$

On simplifying

$$N[U] - \frac{u^\alpha}{s^\alpha} \sum_{k=0}^{\alpha-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} U^{(k)}(0) + \frac{u^\alpha}{s^\alpha} [N[LU] + N[FU] - N[q(x,t)]] = 0 \quad (14)$$

Operating with natural inverse on both sides of (14):

$$U(x,t) = Q(x,t) - N^{-1}[\frac{u^\alpha}{s^\alpha} N[L(U(x,t)) + F(U(x,t))]]. \quad (15)$$

Where $Q(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \quad (16)$$

where the homotopy parameter p is considered as a small parameter $p \in [0,1]$.
We can decompose the nonlinear term as:

$$FU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U), \tag{17}$$

where H_n are He's polynomials of $U_0(x,t), U_1(x,t), U_2(x,t), \ldots, U_n(x,t)$ and it can be calculated by the following formula:

$$H_n(U_0(x,t), U_1(x,t), U_2(x,t), \ldots, U_n(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[F(\sum_{i=0}^{\infty} p^i U_i)]_{p=0}, n = 0,1,2,\ldots \tag{18}$$

By substituting (16) and (17) and using HPM we get:

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = Q(x,t) - p(N^{-1}[\frac{u^\alpha}{s^\alpha} N[L(\sum_{n=0}^{\infty} p^n U_n(x,t)) + (\sum_{n=0}^{\infty} p^n H_n(U(x,t)))]]). \tag{19}$$

This is coupling of natural transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = Q(x,t). \tag{20}$$

$$p^1 : U_1(x,t) = -(N^{-1}[\frac{u^\alpha}{s^\alpha} N[L(U_0(x,t)) + (H_0(U(x,t)))]]), \tag{21}$$

$$p^2 : U_2(x,t) = -(N^{-1}[\frac{u^\alpha}{s^\alpha} N[L(U_1(x,t)) + (H_1(U(x,t)))]]), \tag{22}$$

$$p^3 : U_3(x,t) = -(N^{-1}[\frac{u^\alpha}{s^\alpha} N[L(U_2(x,t)) + (H_2(U(x,t)))]]), \tag{23}$$

Proceeding in the same manner, the rest of the components $U_n(x,t)$ can be completely obtained, and the series solution is thus entirely determined. Finally, we approximate the solution $U(x,t)$ by truncated series.

$$U(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} U_n(x,t). \tag{24}$$

These series solutions generally converge very rapidly.

3. Applications

3.1 Application 1: Sinh-Gordon equation

Sinh-Gordon equation has wide applications in physics and engineering. It appears for example in integrable quantum field [12], fluid dynamics [13], non-commutative field theories [14], kink dynamics ([15-17]) and in many other scientific applications. There is a growing interest in the study of the sinh-Gordon equation ([13, 15, 18-22]) due to the wide applications of sinh-Gordon type equations.
Consider the following time-fractional sinh-Gordon equation

$$D_t^\alpha U - U_{xx} + \sinh(U) = 0; \quad 1 < \alpha \le 2, \quad -\infty < x < \infty, \quad t > 0, \tag{25}$$

Subject to initial condition:

$$U(x,0) = 4\tanh^{-1}(e^{kx})$$

By applying natural transform on both sides of Eq.(25) subject to initial condition, we have

$$N[U] = \frac{u^\alpha}{s^\alpha}[\frac{s^{\alpha-1}}{u^\alpha} U(x,0)] + \frac{u^\alpha}{s^\alpha}[N[U_{xx} - \sinh(U)]] \tag{26}$$

Operating with natural inverse on both sides of (26) we get:

$$U(x,t) = 4\tanh^{-1}(e^{kx}) + N^{-1}[\frac{u^\alpha}{s^\alpha} N[U_{xx} - H(U)]]$$

Now, applying the classical homotopy perturbation technique, the solution can be expressed as a power series in p as given below:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{27}$$

where the homotopy parameter p is considered as a small parameter $p \in [0,1]$.
We can decompose the nonlinear term as

$$FU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U), \tag{28}$$

where H_n are He's polynomials of $U_0(x,t), U_1(x,t), U_2(x,t), \ldots, U_n(x,t)$ and it can be calculated by the following formula:

$$H_n(U_0(x,t),U_1(x,t),U_2(x,t),....,U_n(x,t)) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[F(\sum_{i=0}^{\infty} p^i U_i)]_{p=0} \tag{29}$$

By substituting (28) and (29) and using HPM [27] we get**:**

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = 4\tanh^{-1}(e^{kx}) + pN^{-1}[\frac{u^\alpha}{s^\alpha} N[\partial_{xx}(\sum_{n=0}^{\infty} p^n U_n(x,t)) - H(U)]]$$

This is coupling of natural transform and homotopy perturbation method using He's polynomials. By equating the coefficients of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : U_0(x,t) = 4\tanh^{-1}(e^{kx}). \tag{30}$$

$$p^1 : U_1(x,t) = N^{-1}[\frac{u^\alpha}{s^\alpha} N[U_{0xx} - \sinh(U_0)]],$$

$$U_1(x,t) = [\frac{4k^2 e^{3kx} + 4k^2 e^{kx}}{(1-e^{2kx})^2} - \sinh[4\tanh^{-1}(e^{kx})]]\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{31}$$

$$p^2 : U_2(x,t) = N^{-1}[\frac{u^\alpha}{s^\alpha} N[U_{1xx} - U_1 \sinh(U_0)]],$$

$$U_2 = [[\frac{92k^4 e^{3kx} + 92k^4 e^{5kx} + 4k^4 e^{kx} + 4k^4 e^{7kx}}{(1-e^{2kx})^4} - \sinh[4\tanh^{-1}(e^{kx})]\frac{4ke^{kx}}{1-e^{2kx}}$$
$$-\cosh[4\tanh^{-1}(e^{kx})]\frac{4k^2 e^{kx} + 4k^2 e^{3kx}}{(1-e^{2kx})^2} - \frac{4k^2 e^{3kx} + 4k^2 e^{kx}}{(1-e^{2kx})^2}\sinh[4\tanh^{-1}(e^{kx})] \tag{32}$$
$$+\sinh^2[4\tanh^{-1}(e^{kx})]]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

The approximate solution of Eq. (25) in the series form given by[See figure 1]:

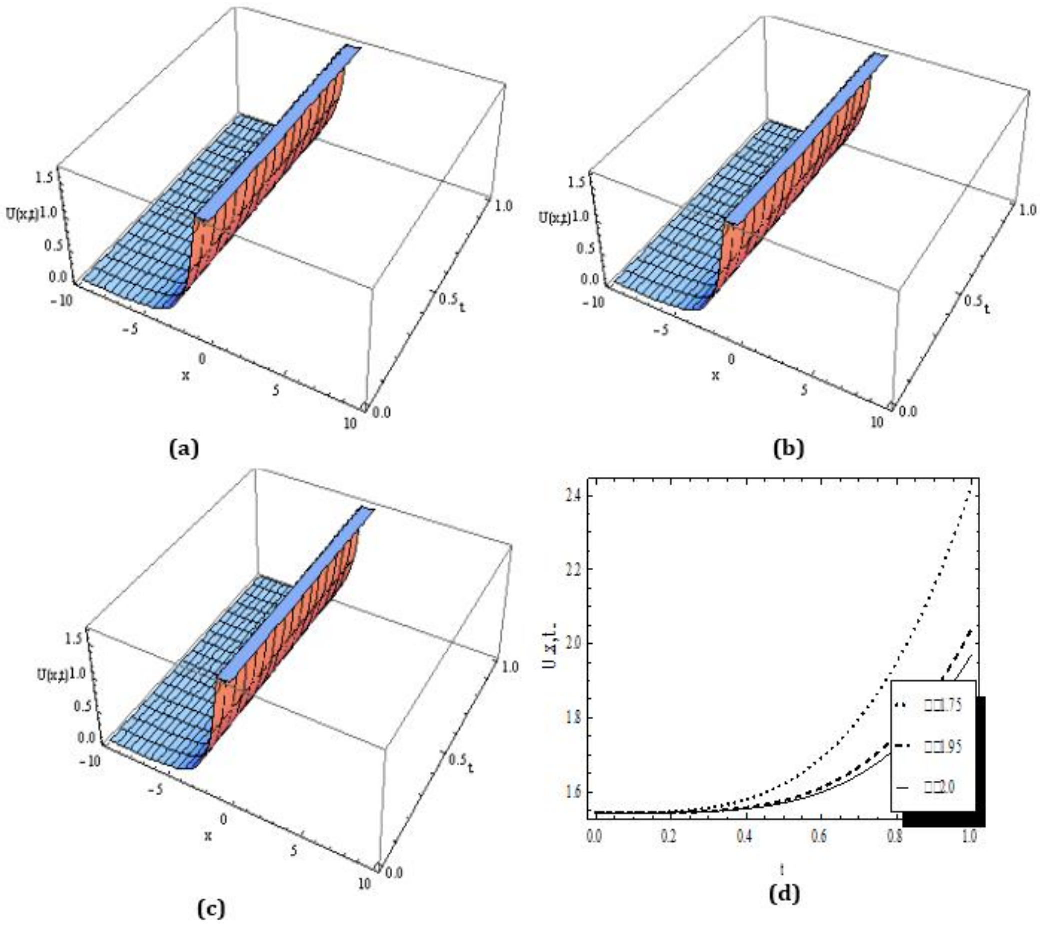


Fig 1: The surface plot of the approximate solution $U(x,t)$ of application 1 when (a) $\alpha = 1.75$, (b) $\alpha = 1.95$, (c) $\alpha = 2$ which is the exact solution, and plot2D of real part of $U(x,t)$ versus t at x =1 for different values of α and comparison the results with the exact solution as shown in (d).

$$U(x,t)=4\tanh^{-1}(e^{kx})+[\frac{4k^2e^{3kx}+4k^2e^{kx}}{(1-e^{2kx})^2}-\sinh[4\tanh^{-1}(e^{kx})]]\frac{t^{\alpha}}{\Gamma(\alpha+1)}+$$
$$[[\frac{92k^{4}e^{3kx}+92k^4e^{5kx}+4k^4e^{kx}+4k^4e^{7kx}}{(1-e^{2kx})^4}-\sinh[4\tanh^{-1}(e^{kx})]\frac{4ke^{kx}}{1-e^{2kx}}$$
$$-\cosh[4\tanh^{-1}(e^{kx})]\frac{4k^2e^{kx}+4k^2e^{3kx}}{(1-e^{2kx})^2}-[\frac{4k^2e^{3kx}+4k^2e^{kx}}{(1-e^{2kx})^2}\sinh[4\tanh^{-1}(e^{kx})] \quad (33)$$
$$+\sinh^2[4\tanh^{-1}(e^{kx})]]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\dots$$

3.2 Application 2: Sharma-Tasso-Olver equation

The nonlinear fractional Sharma-Tasso-Oliver equation is a Korteweg-de Vries-like equation, many physicists and mathematicians have considered it in recent years due to its appearance in scientific applications. The Sharma-Tasso-Olver equation plays an important role in describing the nonlinear wave phenomena. Exact solutions for it with different forms can describe different nonlinear waves. The solution of nonlinear fractional Sharma-Tasso-Olver equation is much involved. In general, there exists no method that yields an exact solution for a nonlinear fractional evolution equation. Only approximate solutions can be derived using the techniques of nonlinear analysis, such as a domian decomposition method (Saha Ray S. *et al*. 2005), homotopy analysis method (Cang J. *et al*. 2009; Liao A S. 1992) and etc. [31].

$$D_t^{\alpha}U(x,t)+\frac{3}{2}a(U^2)_{xx}+a(U^3)_x+aU_{xxx}=0, 0<\alpha\leq 1, t<0 \quad (34)$$

where a is a real parameter, α is a parameter describing the order of fractional time derivative, $U(x,t)$ is an unknown function depending on temporal variable t and spatial variable x.

Subject to the initial condition:

$$U(x,0)=\sqrt{\frac{1}{a}}\tanh(\sqrt{\frac{1}{a}}x) \quad (35)$$

where the exact solution of the Sharma–Tasso–Olver equation with $\alpha=1$ is given by

$$U(x,0)=\sqrt{\frac{1}{a}}\tanh(\sqrt{\frac{1}{a}}(x-t)) \quad (36)$$

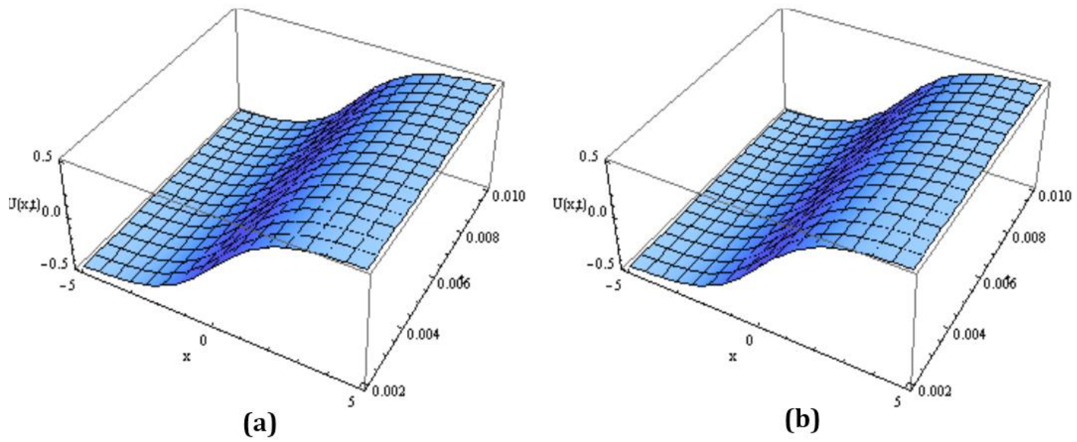
As the previous application, by applying HPNTM method we have

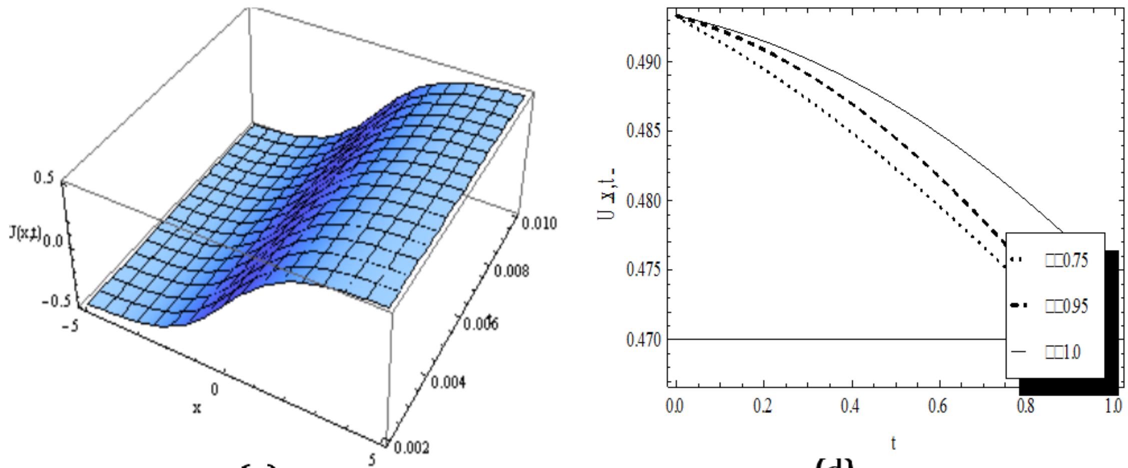
$$U_0(x,t)=\sqrt{\frac{1}{a}}\tanh(\sqrt{\frac{1}{a}}x) \quad (37)$$

$$U_1(x,t)=-\frac{1}{a}\sec h^2(\sqrt{\frac{1}{a}}x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \quad (38)$$

$$U_2(x,t)=-(\frac{1}{a})^{\frac{3}{2}}\sec h^2(\sqrt{\frac{1}{a}}x)\tanh(\sqrt{\frac{1}{a}}x)[(\frac{-12}{a}[\sec h^4(\sqrt{\frac{1}{a}}x)-2\sec h^2(\sqrt{\frac{1}{a}}x)\tanh^2(\sqrt{\frac{1}{a}}x)])+$$
$$(\frac{12}{a^{\frac{1}{2}}}\sec h^2(\sqrt{\frac{1}{a}}x)\tanh(\sqrt{\frac{1}{a}}x))+(8-24\sec h^2(\sqrt{\frac{1}{a}}x))]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\dots \quad (39)$$

The approximate solution of Eq.(25) in the series form given by[See figure 2]:





(c) **(d)**

Fig 2: The surface plot of the approximate solution $U(x,t)$ of application 2 when (a) $\alpha = 0.75$, (b) $\alpha = 0.95$, (c) $\alpha = 1$ which is the exact solution, and plot2D of $U(x,t)$ versus $t$at x =5 for different values of$\alpha$and comparison the results with the exact solution as shown in (d).

$$U(x,t)=\sqrt{\frac{1}{a}}\tanh(\sqrt{\frac{1}{a}}x)-\frac{1}{a}\sec h^{2}(\sqrt{\frac{1}{a}}x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}-(\frac{1}{a})^{\frac{3}{2}}\sec h^{2}(\sqrt{\frac{1}{a}}x)\tanh(\sqrt{\frac{1}{a}}x)$$
$$[(\frac{-12}{a}[\sec h^{4}(\sqrt{\frac{1}{a}}x)-2\sec h^{2}(\sqrt{\frac{1}{a}}x)\tanh^{2}(\sqrt{\frac{1}{a}}x)])+ \tag{40}$$
$$(\frac{12}{a^{\frac{1}{2}}}\sec h^{2}(\sqrt{\frac{1}{a}}x)\tanh(\sqrt{\frac{1}{a}}x))+(8-24\sec h^{2}(\sqrt{\frac{1}{a}}x))]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\ldots$$

4. Conclusion

HPNTM has been utilized to derive the approximate analytical solutions for nonlinear fractional sinh-Gorden equation and fractional Sharma-Tasso-Olver equation. The method gives more realistic series solutions that converge very rapidly in physical problems. To demonstrate the validity of the proposed method, numerical results have been obtained which shows that the HPNTM strength lays in its ease of use and the possibility of using it as a tool to acquire approximate solutions of nonlinear fractional differential equation with excellent accuracy. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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