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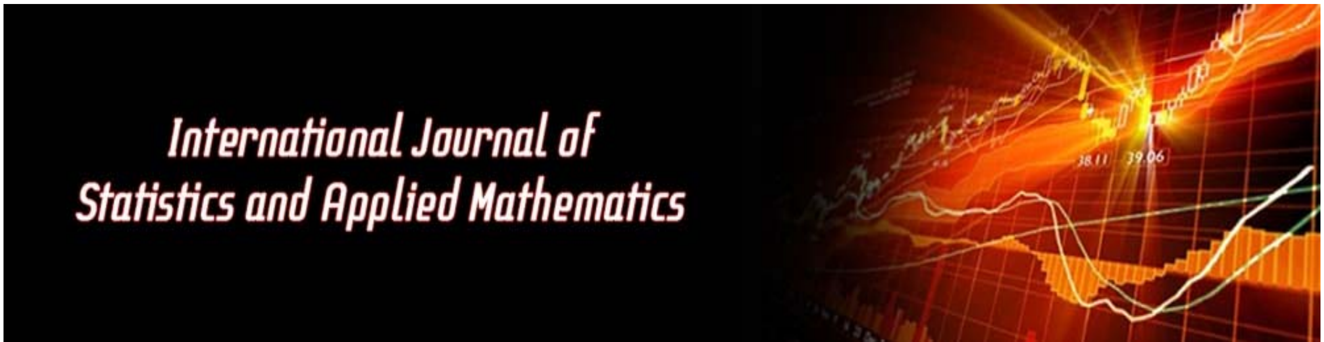

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Multiple solutions for asymptotically linear Schro¨dinger equation with sign-changing potentials

Liao Fangfang, Liu Zhigang and Li Xiaoping

Abstract
In this paper, we study the multiple solutions for a class of asymptotically linear Schr¨odinger equations
$(-\Delta u + V(x)u = f(x,u),\ x \in \mathbb{R}N,\ u(x) \to 0,$ as $|x| \to \infty,$
where the potential $V(x)$ is allowed to be sing-changing, and $f(x,u)$ satisfies the asymptotically linear condition.

Keywords: Multiple solution, Asymptotically linear Shr¨odinger equation, Sing-changing potential, Linking structure

1. Introduction
Consider the following asymptotically linear Schro¨dinger equation

$(-\Delta u + V(x)u = f(x,u)\ x \in \mathbb{R}_N,$ (1.1)

$u(x) \to 0,$ as $|x| \to \infty,$

where $V : \mathbb{R}^N \to \mathbb{R}$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$.

The multiplicity solutions for (1.1) have been extensively investigated in many papers with the aid of variational methods. For instance, see [], [], [], and the references therein.

In the present paper, we will further study the multiplicity solutions for (1.1) where $V(x)$ is allowed to be sign-changing, and $f(x,u)$ satisfies asymptotically linear condition. Our precise assumptions on V and f are the following:

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V > -\infty$;

(V_2) For any $h > 0$, there exists a contant d_0 such that

$$\lim_{|y|\to+\infty} meas\{x \in \mathbb{R}^N : |x-y| \le d_0, V(x) \le h\} = 0,$$

where $meas(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$;

(f_1) $F(x,u) \ge 0$, $F(x,0) = 0$ and $f(x,u) = o(|u|)$, $u \to 0$ uniformly on $\mathbb{R}^N$. $|f(x,u)| \le C_f(|u|)$ for some $C_f > 0$, where $F(x,u) = \int_0^u f(x,s)ds$

It is well known that the Schr¨odinger operator $-\Delta+V$ is selfadjoint and semi bounded in $L^2(\mathbb{R}^N)$. We denote by T the selfadjoint extension of the operator $-\Delta+V$ with domain $D(T) \subset L^2(\mathbb{R})$. Let $\{S(\lambda) : -\infty < \lambda < +\infty\}$ and $|T|$ be the spectral resolution and absolute value of T respectively, and $|T|^{\frac{1}{2}}$ be the square root of $|T|$ with domain $D(|T|^{\frac{1}{2}})$.

Set $U = I - S(0) - S(-0)$, where I is the identity map on L^2. Then U commutes with T, $|T|$ and $|T|^{\frac{1}{2}}$, and $T = U|T|$ is the polar decomposition of T.

Let $S = D(|T|^{\frac{1}{2}})$ and define on S the inner product and norm

$$(u,v)_S = (|T|^{\frac{1}{2}}u, |T|^{\frac{1}{2}}v)_2 + (u,v)_2$$

$$\|u\|_S = (u,u)_S^{\frac{1}{2}}$$

where $(\cdot,\cdot)_2$ denotes the inner product in L^2, then S is a Hilbert space.

In the following, let $\sigma(T)$, $\sigma_d(T)$, $\sigma_e(T)$be the spectrum of T, the discrete spectrum of T and the essential spectrum of T respectively.

In order to learnt about the spectrum fo T, we first need the following lemma.

Lemma 1.1. *Suppose that V satisfies (V_1), (V_2), then the operator T is consists of only eigenvalues numbered in $\lambda_1 < \lambda_2 \le \cdots \le \lambda_n \cdots \le \to \infty$, with a corresponding eigenfunctions $\{e_n\}_{n\in\mathbb{N}} (Te_n = \lambda_n e_n)$, forming an orthogonal basis in L^2.*

Proof. For any $h > 0$, if (V_1) and (V_2) holds. Set

$$(V(x)-h)^+ := \begin{cases} V(x)-h & if\ V(x)-h \ge 0 \\ 0 & if\ V(x) < 0, \end{cases}$$

and $(V(x)-h)^- = (V(x)-h)-(V(x)-h)^+$. Thus $T = T_2 + (V(x)-h)^-$, where
$T_2 = -\Delta + h + (V(x)-h)^+$.
For $u \in D(T)$, we have

$$\begin{aligned}(T_2u, T_2u)_2 &= |(-\Delta + h + (V(x)-h)^+)u|_2^2 \\ &= |(-\Delta + (V(x)-h)^+)u|_2^2 + h^2|u|_2^2 \\ &+ (-\Delta u, hu)_2 + (hu, -\Delta u)_2 \\ &+ ((V(x)-h)^+u, hu)_2 + (hu, (V(x)-h)^+u)_2 \\ &\ge h^2|u|_2^2.\end{aligned}$$

Thus $\sigma(T_2) \subset \mathbb{R}\backslash(-h,h)$.

We claim that $\sigma_e(T) \cap (-h,h) = \emptyset$. Assume by contradiction that there is $v \in \sigma_e(T)$ with $|v| < h$. Let $\{u_n\} \subset D(T)$ with $|u_n|_2 = 1$, $u_n \rightharpoonup 0$ in L^2 and $|(T-v)u_n|_2 \to 0$. Moreover, by (V_0), one can check that the multiplication operator $u \mapsto (V(x)-h)^-u$ is compact. In fact, let $\{u_n\}$ be bounded in $D(T)$, without loss of generality, we may assume $u_n \rightharpoonup 0$ in $D(T)$. Next we show that $|(V(x)-h)^-u_n|_2 \to 0$ in L^2. For every $R > 0$, define $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ and $B_R^c = \mathbb{R}^N\backslash B_R(0)$. Let $\{y_i\}$ be a sequence of points in B_R^c satisfying $B_R^c \subset \bigcup_{i=1}^{\infty} B(y_i, r_0)$ and such that each point x is contained in at most 2^N such balls $B(y_i,r_0)$. Let $B := \{x \in B_R^c : V(x) < h\}$, choose $s \in (2, \frac{N}{N-2})$ and $s' = \frac{s}{s-1}$. We get

$$\begin{aligned}\int_{B_R^c} |(V(x)-h)\ \ u_n|^2 &\le \sum_{i=1}^{\infty} \int_{B(y_i,r_0)\cap B} |(V(x)-h)\ \ u_n|^2 \\ &\le \sum_{i=1}^{\infty} (\int_{B(y_i,r_0)\cap B} |u_n|^{2s})^{\frac{1}{s}} (\int_{B(y_i,r_0)\cap B} |(V(x)-h)^-|^{2s'})^{\frac{1}{s'}} \\ &\le C_R^{2s'} \sum_{i=1}^{\infty} |B(y_i,r_0)\cap B|^{\frac{1}{s'}} (\int_{B(y_i,r_0)\cap B} |u_n|^{2s})^{\frac{1}{s}} \\ &\le C_R^{2s'} \varepsilon_R 2^N \|u_n\|_T^2,\end{aligned}$$

where $C_R = \sup_{y_i} |(V(x)-h)\ \ |, \varepsilon_R = \sup_{y_i} meas(B(y_i,r_0)\cap B)^{\frac{1}{s'}}$. Assumption (V_1) and (V_2) implies that $\varepsilon_R \to 0$ as $R \to \infty$, thus

$$\int_{B_R^c} |(V(x)-h)^- u_n|^2 \to 0.$$

On the other hand,

$$\int_{B_R(0)} |(V(x)-h)^- u_n|^2 \le (\int_{B_R(0)} |u_n|^{2s})^{\frac{1}{s}} (\int_{B_R(0)} |(V(x)-h)^-|^{2s'})^{\frac{1}{s'}} \to 0.$$

Thus $|(V(x)-h)^-u_n|_2 \to 0$, we have

$$\begin{aligned} o(1) = |(T-v)u_n|_2 &= |T_2u_n - vu_n + (V(x)-h)^-u_n|_2 \\ &\ge |T_2u_n|_2 - |v| - o(1) \\ &\ge h - |v| - o(1)\end{aligned}$$

which implies that $0 < h - |v| \le 0$, a contradiction. So $\sigma_e(T) \cap (-h,h) = \emptyset$. Since $h > 0$ is arbitrary it follows that $\sigma(T) = \sigma_d(T)$.

Lemma 1.2: *Suppose that V satisfies (V_1), (V_2), then $S = D(|T|^{\frac{1}{2}})$ is a compactly embeded in $L^p(\mathbb{R}^N)$, for all $2 \le p \le 2^*$, where $2^* = \frac{2N}{N-2}$.*

Proof. In order to show that the embedding $S \hookrightarrow L^p(\mathbb{R}^N)$ is compact for all $p \in [2, 2^*)$, it suffices to prove that $S \hookrightarrow L^2$ is compact. Set $L_k := span\{e_1, \cdots, e_k\}$. Let $P_k: S \to L_k$ denote the orthogonal projection. Consider a weakly converging sequence $u_n \rightharpoonup u$ in S, and define $w_n = u_n - u$ and $K := \sup_n \|w_n\|_S^2$. Given $\varepsilon > 0$ we choose $k \in \mathbb{N}$ so that $\frac{K}{\lambda_k} < \frac{\varepsilon}{2}$. Since $P_kw_n \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\|P_kw_n\|_S^2 < \frac{\varepsilon}{2}$ for all $n \ge n_0$. Therefore, we have

$$|w_n|_2^2 = |P_kw_n|_2^2 + |(I-P_k)w_n|_2^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \ge n_0$. This proves that $u_n \to u$ in L^2. —

Let $m^- = \#\{i|\lambda_i < 0\}$, $m^0 = \#\{i|\lambda_i = 0\}$, and $m = m^- + m^0$.

Set $S^- = span\{e_1,e_2,\cdots,e_{m^-}\}$, $S^0 = span\{e_{m^-+1},\cdots,e_m\} = kerT$ and $S^+ = span\{e_{m+1},\cdots\}$. Then one has the orthogonal decomposition $S = S^- \oplus S^0 \oplus S^+$ with respect to the inner product $(\cdot,\cdot)_0$ on S.
We will introduce on S the following inner product and norm:

$$(u,v) = (|T|^{\frac{1}{2}}u, |T|^{\frac{1}{2}}v)_2 + (u^0,v^0)_2$$

$$\|u\| = (u,u)^{\frac{1}{2}}$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in S = S^- + S^0 + S^+$.

Remark 1.3. *The norms* $\|u\|$ *and* $\|u\|_0$ *on S are equivalent, and for any* $2 \le p \le 2^*$*, there exists* $\beta_p > 0$ *such that*
$\|u\|_p \le \beta_p\|u\|, \forall u \in S.$ (1.2)
where $|\cdot|_p$ *is the norm on* L^p.
Let

$$a(u,v) = (|T|^{\frac{1}{2}}Uu, |T|^{\frac{1}{2}}v)_2, \forall u,v \in S. \tag{1.3}$$

be the quadratic form associated with T. Moreover, using the orthogonal projections with respect to the inner product $(\cdot,\cdot)$, we can define
$a(u,u) = ((p^+ - p^-)u,u) = \|u^+\|^2 - \|u^-\|^2.$ (1.4)
for all $u = u^- + u^0 + u^+ \in S$, where $p^\pm : S \to S^\pm$ are orthogonal projection operator.
We further assume the following hypotheses for $f(x,u)$:
(f_2) $f(x,u) = a(x)u + f_u(x,u)$ with $a(x) \in L^\infty$ is a bounded, continuous real function and $f_u(x,u) = o(|u|)$ as $|u| \to \infty$, $\forall x \in \mathbb{R}^N$;
(f_3) $a_0 := \inf_{x\in\mathbb{R}^N} a(x) > \inf(\sigma(T) \cap (0,\infty))$;
(f_4) $0 \ne \sigma_d(T - a(x))$, where $\sigma_d(T - a(x))$ is the discrete spectrum of $T - a(x)$.
From the above spectral result of the operator T, the set $\sigma(T)\cap(0,a_0)$ consistes of only eigenvalues of finite mulitiplicity, where a_0 is defined in f_3. Let l denote the number of eigenvalues (counted with multiplicity) lying in $(0,a_0)$.

Theorem 1.4: *Suppose that* (V_1),(V_2) *and* (f_1)−(f_4) *are satisfied. Then problem* (1.1) *possesses at least one nontrivial solution. Moreover, if in addition f is odd in u, that is*
(f_5) $f(x,-u) = -f(x,u)$ *for* $x \in \mathbb{R}^N, u \in \mathbb{R}$;
holds, the problem (1.1) *has at least l pairs of nontrival solutions.*

Variational setting and proofs of the main results
For any fixed $a > 0$. Let j be the number of eigenvalues of the operator T (counted with multiplicity) lying in $[-a,a]$. Denote by $f_i(1 \le i \le j)$ the corresponding eigenfunctions and set
$L^{a-} := span\{f_1,\cdots,f_j\}$
then we have the orthogonal decomposition
$L^2 = L^{a-} \oplus L^{a+}, u = u^{a-} + u^{a+}$
where L^{a+} is the orthogonal complement of L^{a-} in L^2. Corresponding, S has the decomposition
$S = S^{a-} \oplus S^{a+}$ with $S^{a-} = L^{a-}$ and $S^{a+} = S \cap L^{a+}$
orthogonal with respect to both the inner products $(\cdot,\cdot)_2$ and $(\cdot,\cdot)$.

Lemma 2.1. *For any fixed* $t > 0$, *let* $S = S^{a-} \oplus S^{a+}$ *as above, then*

$$a|u|_2^2 \le \|u\| \text{ for all } u \in S^{a+} \tag{2.1}$$

where $|\cdot|_2$ *is the norm on* L^2.
Proof. It is obvious from the definition of the norm $\|\cdot\|$ on S and the distribution of the eigenvalues of T.
Now we define a functional ϕ on S by

$$\varphi(u) = \frac{1}{2}a(u,u) - \psi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)dx \tag{2.2}$$

where $F(x,u) = \int_0^u f(x,s)ds$ for all $u = u^- \oplus u^0 \oplus u^+ \in S = S^- \oplus S^0 \oplus S^+$.

Proposition 2.2. *Under assumptions* (V_1), (V_2) *and* (f_1), *the functional* ϕ *is of class* $C^1(S,\mathbb{R})$. *Moreover,*

$$(\varphi'(u),v) = (u^+,v^+) - (u^-,v^-) - \int_{\mathbb{R}^N} f(x,u)v dx \tag{2.3}$$

for all $u = u^- \oplus u^0 \oplus u^+ \in S = S^- \oplus S^0 \oplus S^+$ *and* $v = v^- \oplus v^0 \oplus v^+ \in S = S^- \oplus S^0 \oplus S^+$.
Proof. **Existence of the Gateaux derivative.** Let $u,v \in S$. Given $x \in \mathbb{R}^N$ and $0 < |t| < 1$, using the condition (f_1), by the mean value theorem, there exists $0 < \lambda \in (0,1)$ such that
$|F(x,u(x) + tv(x)) - F(x,u(x))|/|t|$
$= |f(x,u(x) + \lambda tv(x))v(x)|$
$\le C_F(|u(x)| + |v(x)|)|v(x)|.$

The Hölder inequality implies that

$(|u(x)| + |v(x)|)|v(x)| \in L^1$

It follows that the Lebesgue theorem that

$$(\phi'(u), v) = (u^+, v^+) - (u^-, v^-) - \int_{\mathbb{R}^N} f(x,u)v\,dx$$

Continuity of the Gateaux derivative. Assume that $u_n * u$ in S, we will prove that $\phi(u_n) \to \phi(u)$ in $\mathbb{R}^N$. Let $v_n \in C_0^\infty(\mathbb{R}),\ u, v \in S$. Since $C_0^\infty(\mathbb{R})$ is dense in S, so the norms on S and $C_0^\infty(\mathbb{R})$ are equivalent. We obtain, by the Hölder inequality and Theorem $A.4$([?]),

$$\int_{\mathbb{R}^N} (\varphi(u_n) - \varphi(u))|v_n|dx \to 0, \quad \forall v_n \in C_0^\infty(\mathbb{R}).$$

Therefore, $\forall v \in S,\ \exists v_n \in C_0^\infty(\mathbb{R})$, when $v_n \to v$, we obtain that $\|v - v_n\|_S \to 0$.

By $u_n * u$ in S, there exists a constant $M > 0$, such that $|\phi(u_n) - \phi(u)| < M$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} |\phi(u_n) - \phi(u)|v\,dx &= \int_{\mathbb{R}^N} |\phi(u_n) - \phi(u)||v_n + v - v_n|dx \\ &\le \int_{\mathbb{R}^N} |\phi(u_n) - \phi(u)||v_n|dx + \int_{\mathbb{R}^N} |\phi(u_n) - \phi(u)||v - v_n|dx \\ &\le \int_{\mathbb{R}^N} |\phi(u_n) - \phi(u)||v_n|dx + M\|v - v_n\|_S \\ &\to 0.\end{aligned}$$

We say that $\phi \in C^1(S,\mathbb{R})$ satisfie $(PS)-$*condition* if any sequence u_n such that

$$\varphi(u_n) \to c, \varphi'(u_n) \to 0, \text{ as } n \to \infty. \quad (2.4)$$

has a convergent subsequence.

Lemma 2.3. *([13], Theorem 5.3)Let S be a real Banach space with $S = Y \oplus Z$, where Y is finite dimensional. Suppose $\phi \in C^1(S,\mathbb{R})$, satisfies $(PS)-$condition, and*

(ϕ_1) there are constants $\rho, \alpha > 0$ such that $\phi|_{\partial B_\rho \cap Z} \ge \alpha$, and

(ϕ_2) there is an $e \in \partial B_\rho \cap Z$ and $\mathrm{R} \ge \rho$ such that if $Q \equiv (\overline{B_\mathrm{R} \cap Y}) \oplus \{re | 0 < r < \mathrm{R}\}$, then $\phi|_{\partial Q} \le 0$. where B_r is an open ball in S of radius r centered at 0.

Then ϕ possess a critical value $c \ge \alpha$ which can be characterized as

$$c \equiv \inf_{h \subset \Gamma} \max_{u \subset Q} \varphi(h(u)),$$

where

$\Gamma = \{h \in C(\overline{Q}, S) | h = id \text{ on } \partial Q\}$.

The other one is the Z2-symmetric Mountain Pass Theorem:

Lemma 2.4. *([10, Corollary 7.22])Let ϕ be an even C^1-function satisfying $(PS)-$ condition on $S = Y \oplus Z$, where $\dim(Y) = k < \infty$. Assume $\phi(0) = 0$ as well as the following conditions:*

(1) *there is $\rho > 0$ and $\alpha \ge 0$ such that* $\inf \phi(S_\rho(Z)) \ge \alpha$.

(2) *there exits $\mathrm{R} \ge \rho$ and a subspace G of S containing Y such that $\dim(G) = n > k$ and* $\sup \phi(S_\mathrm{R}(G)) \le 0$.

Then exists the critical values $c_i (1 \le i \le n - k)$ for ϕ such that (a) $0 \le \alpha \le c_1 \le \cdots \le c_{n-k}$.

(b) *ϕ has at least $n - k$ distinct pairs of non-trivial critical points.*

Lemma 2.5. *Under assumptions (V_1), (V_2), (f_1), (f_2) and (f_4), any sequence $u_n \subset S$ satisfying*

$$\phi(u_n) \to c, \phi'(u_n) \to 0, \quad \text{as } n \to \infty$$

is bounded.

Proof. Arguing indirectly we assume that, up to a subsequence, $\|u_n\| \to \infty$ and set $v_n = u_n/\|u_n\|$, then $v_n = 1$. By lemma 1.2, $v_n * v$ in S and $v_n \to v$ in L^p for all $2 \le p \le 2^*$, then v_n is bounded in L^{2^*}. Since, by (f_1) and (f_2), $|f(x,u)| \le C_F(|u|)$ for some C_F, $f_u(x,u) = o(|u|)$ as $|u| \to \infty$, $x \in \mathbb{R}^N$, if $v \ne 0$, then it follows, by Lebesgue's Dominated Convergence Theorem, that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{(f(x,u_n),\psi)}{\|u_n\|}dx &= \int_{\mathbb{R}^N} \frac{(a(x)u_n,\psi)}{\|u_n\|}dx + \int_{\mathbb{R}^N} \frac{(f_u(x,u_n),\psi)}{\|u_n\|}dx \\ &= \int_{\mathbb{R}^N} (a(x)v_n,\psi)dx + \int_{\mathbb{R}^N} \frac{(f_u(x,u_n),\psi)|v_n|}{|u_n|}dx \\ &\to \int (a(x)v_n,\psi)dx \text{ as } n \to \infty\end{aligned}$$

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for all $\psi \in C_0^\infty(\mathbb{R}^N)$). By (2.3), we have

$$\frac{\varphi'(u_n)\psi}{||u_n||} - (v_n^+, \psi) - (v_n^-, \psi) - \int_{\mathbb{R}^N} \frac{(f(x,u_n),\psi)}{||u_n||} dx$$

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for all $\psi \in C_0^\infty(\mathbb{R}^N)$). From this we deduce, using (2.4), that

$(-\Delta + V)v(x) = a(x)v(x)$ i.e.

$(-\Delta + V - a(x))v(x) = 0$

we claim that $v \neq 0$. Arguing by contradiction we assume that $v = 0$, choose $a > 0$ in lemma 2.1, such that $\frac{C_F}{a} < 1$, where C_F is the constant in (f_1).

Since $S^{a-} \subset S$ in lemma 2.1 of finite-dimension, then the compactness of the orthogonal projection $P^{a-} :\to S^{a-} \subset S$ implies $v_n^{a^-} \to v^{a^-} = 0$ in S.

It follows from (2.4) that

$$\frac{\varphi'(u_n)((u_n^{a^+})^+ - (u_n^{a^+})^-)}{||u_n||^2} = ||v_n^{a^+}||^2 - \int_{u_n \neq 0} \frac{(f(x,u_n),(v_n^{a^+})^+ - (v_n^{a^+})^-)}{|u_n|}|v_n|dx$$

then

$$\begin{aligned}
||v_n^{a^+}||^2 &= \int_{u_n \neq 0} \frac{(f(x,u_n),(v_n^{a^+})^+ - (v_n^{a^+})^-)}{|u_n|}|v_n|dx + \frac{\varphi'(u_n)((u_n^{a^+})^+ - (u_n^{a^+})^-)}{||u_n||^2} \\
&\leq C_F \int_{\mathbb{R}^N} |(v_u^{a^+})^+ - (v_n^{a^+})^-||v_n|dx + \frac{||\varphi'(u_n)||}{||u_n||} \\
&\leq \frac{C_F}{2}\left(\int_{\mathbb{R}^N} |(v_n^{a^+})^+ + (v_n^{a^+})^-|^2 dx + \int_{\mathbb{R}^N} |(v_n^{a^+})^+ - (v_n^{a^+})^-|^2 dx\right) \\
&\quad + \frac{C_F}{2}\int_{\mathbb{R}^N} |v_n^{a^-}|^2 dx + \frac{||\varphi'(u_n)||}{||u_n||} \\
&= C_F |v_n^{a^+}|_2^2 + \frac{C_F}{2}|v_n^{a^-}|_2^2 + \frac{||\varphi'(u_n)||}{||u_n||} \\
&\leq \frac{C_F}{a}||v_n^{a^+}||^2 + \frac{C_F}{2}|v_n^{a^-}|_2^2 + \frac{||\varphi'(u_n)||}{||u_n||}
\end{aligned}$$

where $|\cdot|_2$ is the norm on L^2 and $(\cdot)^+$, $(\cdot)^-$ are the respective components with respect to the orthogonal decomposition. The last inequality follows by lemma

2.3. Note that $v_n^{a^-} \to 0$ in L^2 since $v_n^{a^-} \to 0$ in S. Thus $\frac{C_F}{2} < 1$ and (2.4) imply

$||v_n^{a^+}||^2 \to 0$. Then $1 = ||v_n||^2 = ||v_n^{a^-}||^2 + ||v_n^{a^-}||^2 \to 0$, a contradiction.

Therefore, $v \neq 0$. Then 0 is an eigenvalue of $A-a(x)$ which is in contradiction to (f_4).

Lemma 2.6. *Suppose that* (f_1), (f_2) *and* (f_4) *are satisfied. Then* ϕ *satisfies the* (*PS*)*-condition.*

Proof. Let $u_n \subset S$ be an arbitrary (*PS*)*−sequence*. By lemma 2.5, it is bounded, hence, we may assume without loss of generality that $u_n * u$ in S and hence $u_n^+ \rightharpoonup u^+$ and $u_n^- \to u^-$ due to $dim(S^-) < \infty$. By lemma 1.2, $u_n \to u$ and

$u_n^+ \to u^+$ in L^2. Observe that

$||u+n - u+m||2 = (\phi 0(un) - \phi 0(um))(u+n - u+m)$

Z (2.5)

$+ \quad (f(x,u_n) - f(x,u_m), u_n^+ - u_m^+)dx \ \forall n,m \in \mathbb{N}.$

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by (f_1) and Ho¨lder inequality

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$|(f(x,u_n) - f(x,u_m), u_n^+ - u_m^+)dx| \leq \quad C_f \quad (|u_n| + |u_m|)|u_n^+ - u_m^+|dx$

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$\leq Cf(|un|2 + |um|2)|u+n - u+m|2dx$ as $n,m \to \infty$

since $u_n \to u$ and $u_n^+ \to u^+$ in L^2.

Note that

$(\varphi'(u_n) - \varphi'(u_m))(u_n^+ - u_m^+) \to 0$ as $n,m \to \infty$

since $\phi^0(u_n) \to 0$ and (u_n) is bounded in S. Then (u_n^+) is a Cauchy sequence in S. Hence $u_n^+ \to u^+$ in S. Recall that $dim(S^- \oplus S^0) < \infty$, then $u_n^- + u_n^0 \to u^- + u^0$ in S. This yields $u_n \to u$ in S and the proof is completed.

Lemma 2.7. *Let* (f_1) *be satisfied, then there exits* $\rho > 0$ *such that*
$\alpha := \inf \phi(\partial B_\rho \cap S^+) > 0$
Proof. By lemma 1.2, we have
$|u|_\infty \to 0$ as $\|u\| \to 0$ (2.6)
where $|\cdot|_\infty$ is the norm on L^∞. From (f_1), we obtain that $F(x,u) = o(|u|^2)$ as $|u| \to 0$ uniformly in x. Therefore, for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\int_{\mathbb{R}^N} F(x,u)dx \le \varepsilon|u|_2^2 \le \beta_2^2\|u\|^2, \quad \forall \|u\| \le \delta$$

where β_2 is the constant. Taking $\varepsilon = 1/(\beta_2^2)$ and $0 < \rho < \delta$, then by the form of ϕ we have $\inf \phi(\partial B_\rho \cap S^+) \ge \rho^2/4 > 0$.
Due to (f_3) and the spectral result of T in the previous section, we can arrange all the eigenvalues (counted with multiplicity) of T in $(0,a_0)$ by $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_l < a_0$ and let e_j denote the corresponding eigenfunctions: $Te_j = \lambda_j e_j$ for $j = 1,\cdots,l$. Set $S_l^+ := span\{e_1,\cdots,e_l\}$. According to the definition of the norm on S, we have

$$\lambda_1|v|_2^2 \le \|v\|^2 \le \lambda_l|v|_2^2, \ for all v \in S_l^+. \tag{2.7}$$

Set $\tilde{S} = S^- \oplus S^0 \oplus S_l^+$.

Lemma 2.8. *Let* (f_1), (f_2) *and* (f_3) *be satisfied and* $\rho > 0$ *be given by Lemma 2.7. Then there exists* $R_{\tilde{S}} > \rho$ *such that* $\phi(u) < 0$ *for all* $u \in \tilde{S}$ *with* $\|u\| \ge R_{\tilde{S}}$
Proof. It suffice to show that $\phi(u) \to -\infty$ as $u \in \tilde{S}$, $\|u\| \to \infty$. Arguing indirectly we assume that there exist some $c > 0$ and a sequence $(u_n) \subset \tilde{S}$ such that $\phi(u_n) \ge -c$ for all n. Denote $v_n := u_n/\|u_n\|$, we have $\|v_n\| = 1$, and passing to a subsequence if necessary, $v_n \to v$, $v_n^- \to v^-$, $v_n^0 \to v^0$ and $v_n^+ \to v^+$ since $\dim(\tilde{S}) < \infty$. Now we have

$$\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}dx = \frac{\varphi(u_n)}{\|u_n\|^2} \ge \frac{-c}{\|u_n\|^2}, \tag{2.8}$$

We claim that $v^+ \neq 0$. Since $F(x,z) \ge 0$, if $v^+ = 0$, it follows from (2.8) and (f_1) that $\|v_n^-\| \to 0$ and thus $v_n \to v = v^0$. Also $\int_{\mathbb{R}^N} \frac{f(x,u_n)}{\|u_n\|^2}dx \to 0$, and $v_n \to v$ in L^2.
Note that by (f_1) and (f_2), $F(x,u) = \frac{1}{2}a(x)u \cdot u + f(x,u)$ and $\|f(x,u)\| \le C_f\|u\|^2$ for some $C_f > 0$, $f(x,u)/\|u\|^2 \to 0$ as $\|u\| \to \infty$, $\forall x \in \mathbb{R}^N$. Since $\|u(x)\| \to \infty$ if $v(x) \neq 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|f(x,u_n)|}{\|u_n\|^2}dx &= \int_{u_n\neq 0} \frac{|f(x,u_n)|}{|u_n|^2}|v_n|^2dx \\ &\le 2\int_{u_n\neq 0} \frac{|f(x,u_n)|}{|u_n|^2}|v_n - v|^2dx + 2\int_{u_n\neq 0} \frac{|f(x,u_n)|}{|u_n|^2}|v|^2dx \\ &\le 2C_f\int_{u_n\neq 0} |v_n - v|^2dx + 2\int_{u_n\neq 0} \frac{|f(x,u_n)|}{|u_n|^2}|v|^2dx \\ &= o(1)\end{aligned} \tag{2.9}$$

The last equality holds by $v_n \to v$ in L^2 and Lebesgue's Dominated Convergence Theorem. Also, by (f_3),

$$\frac{1}{2}\int_{\mathbb{R}^N} \frac{|(a(x)u_n,u_n)|}{\|u_n\|^2}dx = \frac{1}{2}\int_{u_n\neq 0} \frac{|(a(x)u_n,u_n)|}{|u_n|^2}|v_n|^2dx \ge \frac{a_0}{2}|v_n|_2^2. \tag{2.10}$$

From 2.9, 2.10 and since $\int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|^2}dx \to 0$ it follows that $|v_n|_2 \to 0$. Due to $\dim(\tilde{S}) < \infty$, $1 = \|v_n\| \to 0$ and this contradiction implies that $v^+ \neq 0$. Note that (f_3) and (**??**) implies that

$$\|v^+\|^2 - \|v^-\|^2 - \int_{\mathbb{R}N} (a(x)v,v)dx \le \|v^+\|^2 - \|v^-\|^2 - a_0|v|_2^2$$
$$\le -((a_0 - \lambda_l)|v^+|_2^2 + \|v^-\|^2 + a_0|v^- + v^0|_2^2) < 0.$$

Then there exists $I > 0$ such that

$$\int_{-B}^{B} \|v^+\|^2 - \|v^-\|^2 - (a(x)v,v)dx < 0. \tag{2.11}$$

By **??**, we get

$$\lim_{n\to\infty}\int_{-I}^{I}\frac{f(x,u_n)}{\|u_n\|^2}dx \to 0$$

Thus (2.8)-(2.11) imply that

$$0 \le \lim_{j\to\infty}\left(\frac{1}{2}(\|v_j^+\|^2 - \|v_j^-\|^2) - \int_{-I}^{I}\frac{f(x,u_n)}{\|u_n\|^2}dx\right)$$
$$\le \frac{1}{2}(\|v^+\|^2 - \|v^-\|^2 - \int_{-I}^{I}(a(x)v,v)dx) < 0.$$

Now the desired conclusion is obtained from this contradiction.
As an immediate result of Lemma 2.8, we get

Lemma 2.9. *Let* (f_1), (f_2) *be satisfied and* $\rho > 0$ *be given by Lemma 2.7. Then* $e \in S_l^+$ *with* $\|e\| = 1$, *there exists* $R > \rho$ *such that* $\sup\phi(\partial Q) \le 0$ *where* $Q := \{u = u_1 + re : u_1 \in S^- \oplus S^0, \|u_1\| \le R, 0 < r < R\}$.
Proof. Set $R = R_{\tilde{S}}$, where $R_{\tilde{S}}$ is the constant in Lemma 2.8. Then, by Lemma
2.8

$\phi(u) < 0, \forall u \in S^- \oplus S^+ \oplus span\{e\} \subset \tilde{S}, \|u\| \ge R.$ (2.12)

Observe that
$\partial Q = Q_1 \cup Q_2 \cup Q_3$,
where
$Q_1 := \{u \in S^- \oplus S^0, \|u\| \le R\}$,
$Q_2 := \{u = u_1 + Re : u_1 \in S^- \oplus S^0, \|u_1\| \le R\}$,
$Q_3 := \{u = u_1 + re : u_1 \in S^- \oplus S^0, \|u_1\| = R, 0 < r < R\}$.
Due to (2.12), it holds that
$\phi(u) \le 0, \forall u \in Q_2 \cup Q_3$
Aslo, in view of (f_1) and the form of ϕ in 2.2, $\phi(u) \le 0, \forall u \in Q_1$. Then the proof is completed.
After all the above preparations, we now come to the proof of our main result.
Proof.
Proof of Theorem 1.4. *Existence*. With $Y = S^- \oplus S^0$ and $Z = S^+$ in Theorem 2.3, the condition (ϕ_1) holds by Lemma 2.7 and (ϕ_2) holds by Lemma 2.9. Lemma 2.7 shows that ϕ satisfies the $(PS) - condition$. Hence, ϕ has at least one critical point u with $\phi(u) \ge \alpha > 0$ by Theorem 2.3. Since $\phi(0) = 0$, u is a nontrvial critical point of ϕ. Then (1.1) has at least one nontrivial solution u by Proposition 2.2.
Multiplicity. Let $Y = S^- \oplus S^0$ and $Z = S^+$ in Theorem 2.4. Since $F(x,u)$ is even in u, then ϕ is even and $\phi(0) = 0$ by the form of ϕ in (2.2). Lemma 2.7 shows that (1) in Theorem 2.4 holds. With $G = \tilde{S}$ in Theorem 2.4, then Lemma 2.8 implies that (2) in Theorem 2.4 also holds. Note that $\dim(G) - \dim(Y) = \dim(S_l^+ = l)$. Therefore, ϕ has at least l pairs of nontrivial critical points by Theorem 2.4 and then (1.1) has at least l pairs of nontrivial solution u by Proposition 2.2. □

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