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Lower and upper bound for parametric 'Useful' R-norm information measure

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Abstract
A parametric mean length is defined as the quantity

$$ {}_{R\beta}L_u = \frac{R}{R-1}\left[1-\left(\frac{\sum u_i\, p_i^{\beta} D^{-n_i(R-1)}}{\sum u_i\, p_i^{\beta}}\right)^{\frac{1}{R}}\right], $$

Where $R > 0\ (\neq 1),\ \beta > 0, u_i > 0, D > 1$ is an integer, $\sum p_i = 1$.This being the useful mean length of code words weighted by utilities, u_i. Lower and Upper bounds for ${}_{R\beta}L_u$ are derived in terms of 'useful' R-norm information measure.

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1. Introduction

Consider the following model for a random experiment S,

$S_N = [E;\ P;\ U]$

Where $E = (E_1,\ E_2,\ \ldots,\ E_N)$ is a finite system of events happening with respective probabilities $P = (p_1,\ p_2,\ \ldots,\ p_N)$, $p_i \geq 0$, $\sum p_i = 1$ and credited with utilities $U = (u_1,\ u_2,\ \ldots,\ u_N)$, $u_i > 0,\ i = 1,\ 2, \ldots,\ N$. Denote the model by S_N, where,

$$ S_N = \begin{bmatrix} E_1, E_2, \ldots, E_N \\ p_1, p_2, \ldots, p_N \\ u_1, u_2, \ldots, u_N \end{bmatrix} \tag{1.1} $$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [3] proposed a measure of information called 'useful information' for this scheme, given by

$$ H(U;\ P) = -\sum u_i\ p_i \log\ p_i, \tag{1.2} $$

Where $H(U;\ P)$ reduces to Shannon's [16] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each *i*. throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D\ (D > 1)$.

Guiasu and Picard [5] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1,\ w_2,\ \ldots,\ w_N$ having lengths $n_1,\ n_2,\ \ldots,\ n_N$ and satisfying Kraft's inequality [4].

$$\sum_{i=1}^{N} D^{-n_i} \leq 1 \quad . \tag{1.3}$$

Where D is the size of the code alphabet. The useful mean length L_u of code was defined as:

$$L_u = \frac{\sum u_i\, n_i\, p_i}{\sum u_i\, p_i} \quad , \tag{1.4}$$

and the authors obtained bounds for it in terms of $H(U;\, P)$. Generalized coding theorems by considering different generalized measures under condition (1.3) of unique decipherability were investigated by several authors, see for instance the papers [7, 8, 11, 12, 13, 17].

In this paper, we study some coding theorems by considering a new function depending on the parameters R, β and a utility function. Our motivation for studying this new function is that it generalizes 'useful' R-norm information measure already existing in the literature such as Boekee and Lubbe [2] and Satish [11,13].

2. Coding Theorems

In this section, Satish [11] studied 'useful' R-norm information measure as:

$$_{R\beta}H\left(U;\, P\right) = \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i\, p_i^{R\beta}}{\sum u_i\, p_i^{\beta}}\right)^{\frac{1}{R}}\right], \tag{2.1}$$

where $\beta > 0, R > 0\ (\neq 1),\ u_i > 0,\ p_i \geq 0,\ i = 1, 2, \ldots, N$ and $\sum p_i = 1$.

(i) If $\beta = 1$, then (2.1) becomes 'useful' R-norm information measure studied by Singh and Rajeev [18].

(ii) When $u_i = 1$ for each i, i.e., when the utility aspect is ignored, $\sum p_i = 1$, and $\beta = 1$, then (2.1) reduces to R-norm entropy as considered by Boekee and Lubbe [2].

i.e.,
$$_{R}H\left(P\right) = \frac{R}{R-1}\left[1 - \left(\sum p_i^{R}\right)^{\frac{1}{R}}\right]. \tag{2.2}$$

(iii) When $R \to 1$, and $\beta = 1$, then (2.1) reduces to a measure of 'useful' information due to Hooda and Bhaker [1].

i.e.,
$$H\left(U;P\right) = -\frac{\sum u_i p_i \log p_i}{\sum u_i p_i} \quad . \tag{2.3}$$

(iv) When $u_i = 1$ for each i, i.e., when the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$, and $R \to 1$, the measure (2.1) reduces to Shannon's entropy [16].

i.e.,
$$H\left(P\right) = -\sum p_i \log p_i \quad . \tag{2.4}$$

(v) If $u_i = 1$, then (2.1) reduces to R-norm entropy for power of probability distribution p_i^{β}

i.e.,
$$_{R\beta}H\left(P\right) = \frac{R}{R-1}\left[1 - \left(\frac{\sum p_i^{R\beta}}{\sum p_i^{\beta}}\right)^{\frac{1}{R}}\right]. \tag{2.5}$$

Further consider

Definition: The 'useful' mean length $_{R\beta}L_u$ with respect to 'useful' R-norm information measure is defined as:

$$_{R\beta}L_u = \frac{R}{R-1}\left[1 - \left(\frac{\sum u_i p_i^{\beta} D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}}\right)^{1/R}\right], \tag{2.6}$$

Under the condition,
$$\sum u_i D^{-n_i R} \leq \sum u_i\, p_i^{R\beta} \quad . \tag{2.7}$$

Clearly the inequality (2.7) is the generalization of Kraft's inequality (1.3). A code satisfying (2.7) would be termed as a useful personal probability code D (D>2) is the size of the code alphabet. When, $u_i = 1$ for each i and $\beta = 1, R = 1$, (2.7) reduces to (1.3).

(i) For $u_i = 1$ for each i and $\beta = 1$, and $R \to 1$, $_{R}L_u$ becomes the optimal code length defined by Shannon [16].

(ii) For $u_i = 1$ for each i and $\beta = 1$, then (2.6) reduced to ${}_R L$ considered by Boekee and Lubbe [2] and Satish and Arun [13].

i.e., $${}_R L = \frac{R}{R-1}\left[1-\left(\sum p_i D^{-n_i(R-1)}\right)^{1/R}\right]. \quad (2.8)$$

(iii) For $u_i = 1$ for each i, then (2.6) reduces to mean code word length corresponding to the entropy (2.5)

i.e., $${}_{R\beta} L = \frac{R}{R-1}\left[1-\left(\frac{\sum p_i^{\beta} D^{-n_i(R-1)}}{\sum p_i^{\beta}}\right)^{1/R}\right]. \quad (2.9)$$

(iv) For $\beta = 1$, then (2.6) becomes

$${}_R L_u = \frac{R}{R-1}\left[1-\left(\frac{\sum u_i p_i D^{-n_i(R-1)}}{\sum u_i p_i}\right)^{1/R}\right], \quad (2.10)$$

which is a useful R-norm mean codeword length.

We establish a result, that in a sense, provides a characterization of ${}_{R\beta} H(U;\ P)$ under the condition of unique decipherability.

Theorem 2.1. Let $u_i, p_i, n_i,\ i = 1,2,...,N,$ satisfy the inequality (2.7). Then

$${}_{R\beta} L_u \geq {}_{R\beta} H(U;P), \quad 1 \neq R > 0, \beta > 0. \quad (2.11)$$

Proof: By Holder's inequality, we have

$$\left(\sum_{i=1}^{N} x_i^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} y_i^{q}\right)^{\frac{1}{q}} \leq \sum_{i=1}^{N} x_i y_i, \quad (2.12)$$

where $p^{-1} + q^{-1} = 1;\ p(\neq 0) < 1, q < 0$ or $q(\neq 0) < 1,\ p < 0;\ x_i, y_i > 0$ for each i.

Setting, $p = \frac{(R-1)}{R}, q = 1 - R$ and

$$x_i = \left(\frac{u_i p_i^{\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{R}{R-1}} D^{-n_i R}, \quad y_i = \left(\frac{u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R-1}} \quad (2.13)$$

Putting these values in (2.12) and using the inequality (2.7), we get

$$\left(\frac{\sum u_i p_i^{\beta}\ D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}}\right)^{\frac{R}{R-1}} \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R-1}} \leq \frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}} \quad (2.14)$$

It implies

$$\left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{R}{1-R}} \leq \left(\frac{\sum u_i p_i^{\beta}\ D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}}\right)^{\frac{R}{R-1}} \quad (2.15)$$

Now consider two cases:

Case 1: Let $0 < R < 1$. Raising both sides of (2.15) to the power $(1-R)/R^2$, we get

$$\left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R}} \leq \left(\frac{\sum u_i p_i^{\beta}\ D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{R}} \quad (2.16)$$

Since $R/(1-R) > 0$ for $0 < R < 1$, we get from (2.16) the inequality (2.11).

Case 2: Let $R > 1$. The proof follows on the same lines.

It is clear that the equality in (2.11) is true if and only if

$$D^{-n_i} = p_i^{\beta}$$

which implies that

$$n_i = \log_D \frac{1}{p_i^{\beta}} \tag{2.17}$$

Thus, it is always possible to have a codeword satisfying the requirement

$$\log_D \frac{1}{p_i^{\beta}} \le n_i < \log_D \frac{1}{p_i^{\beta}} + 1,$$

which is equivalent to

$$\frac{1}{p_i^{\beta}} \le D^{n_i} < \frac{D}{p_i^{\beta}}. \tag{2.18}$$

In the following theorem, we give an upper bound for ${}_{R\beta}L_u$ in terms of ${}_{R\beta}H(U;P)$.

Theorem 2.2. By properly choosing the lengths $n_1, n_2, ..., n_N$ in the code of Theorem 2.1, ${}_{R\beta}L_u$ can be made to satisy the following inequality:

$$ {}_{R\beta}L_u < D^{(1-R)/R}\ {}_{R\beta}H(U;P) + \frac{R}{R-1}(1 - D^{(1-R)/R}) \tag{2.19}$$

Proof: From (2.18), it is clear that

$$D^{-n_i} > D^{-1} p_i^{\beta}. \tag{2.20}$$

We have again the following two possibilities.

(i) Let $R > 1$. Raising both sides of (2.20) to the power $(R-1)$, we have

$$D^{-n_i(R-1)} > D^{1-R} p_i^{\beta(R-1)}.$$

Multiplying both sides by $u_i p_i^{\beta}$ and then summing over $1\ to\ N$. we get

$$\sum u_i p_i^{\beta} D^{-n_i(R-1)} > D^{1-R} \sum u_i p_i^{R\beta}. \tag{2.21}$$

Obviously (2.21) can be written as

$$\left[\frac{\sum u_i p_i^{\beta} D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}}\right]^{\frac{1}{R}} > D^{\frac{(1-R)}{R}} \left[\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}\right]^{\frac{1}{R}}. \tag{2.22}$$

Since $R - 1 > 0$ for $R > 1$, we get the inequality (2.19) from (2.22).

(ii) If $0 < R < 1$, the proof follows similarly. But the inequality (2.22) is reversed.

Theorem 2.3. For arbitrary $N \in \mathbb{N}, 1 \ne R > 0, \beta > 0,$ and for every codeword lengths $n_i, i = 1, 2, ..., N$ of Theorem 2.1, ${}_{R\beta}L_u$ can be made to satisy the following inequality:

$$ {}_{R\beta}L_u \ge {}_{R\beta}H(U;P) > D^{1/R}\ {}_{R\beta}H(U;P) + \frac{R}{R-1}\left(1 - D^{1/R}\right). \tag{2.23}$$

Proof: Suppose,

$$\overline{n}_i = \log_D \frac{1}{p_i^{\beta}}, \quad \beta > 0. \tag{2.24}$$

Clearly $\overline{n}_i$ and $\overline{n}_i + 1$ satisfy the equality in Holder's inequality (2.12). Moreover, $\overline{n}_i$ satisfies (2.7). Suppose $\overline{n}_i$ is the unique integer between $\overline{n}_i$ and $\overline{n}_i + 1$, then obviously, $\overline{n}_i$ satisfies (2.7).

Since $1 \ne R > 0$, $\beta > 0$, we have

$$\frac{\sum u_i p_i^{\beta} D^{-n_i(R-1)}}{\sum u_i p_i^{\beta}} \le \frac{\sum u_i p_i^{\beta} D^{-\overline{n}_i(R-1)}}{\sum u_i p_i^{\beta}}$$

$$< D\left(\frac{\sum u_i p_i^{\beta} D^{-\overline{n}_i(R-1)}}{\sum u_i p_i^{\beta}}\right). \tag{2.25}$$

Since, $\frac{\sum u_i p_i^{\beta} D^{-\bar{n}_i (R-1)}}{\sum u_i p_i^{\beta}} = \frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}}$.

Hence (2.25) becomes

$$\frac{\sum u_i p_i^{\beta} D^{-n_i (R-1)/R}}{\sum u_i p_i^{\beta}} \le \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}} \right)^{\frac{1}{R}} < D^{1/R} \left(\frac{\sum u_i p_i^{R\beta}}{\sum u_i p_i^{\beta}} \right)^{\frac{1}{R}}.$$

Which gives (2.23).

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