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The generalized hypercenter and the structure of finite groups

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Abstract
We study the structure of a finite group G under the assumption that certain subgroups lie in the generalized hypercenter of G. Our results generalize some well-known results.
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1. Introduction

All groups considered in this paper will be finite. For a p-group P, we denote $\Omega(P)=\Omega_1(P)$ if p>2 and $\Omega(P)=\langle\Omega_1(P),\Omega_2(P)\rangle$ if p=2, where $\Omega_i(P)=\langle x\in P \mid o(x)=p^i\rangle$. Two subgroups H and K of a group G are said to permute if HK =KH. It is easily seen that H and K permute if and only if the set HK is a subgroup of G. A subgroup of G is quasinormal in G if it permutes with every subgroup of G. We say, following Kegel [7], that a subgroup of G is S-quasinormal in G if it permutes with every Sylow subgroup of G. Agrawal [1] defined the generalized center, genz (G), of G to be the subgroup generated by all elements g of G such that $\langle g\rangle$ is S-quasinormal in G. The generalized hypercenter, genz_∞(G), is the largest term of the series $1=\text{genz}_0(G)\leq \text{genz}_1(G)=\text{genz}(G)\leq \text{genz}_2(G)\leq \dots$, where $\text{genz}_{i+1}(G)/\text{genz}_i(G)=\text{genz}(G/\text{genz}_i(G))$ for all $i\geq 0$. More recently, Asaad and Ezzat [3] have given a new characterization of genz_∞(G) by introducing the following definition: A normal subgroup H of a group G is generalized supersolvably embedded, GSE, in G if there exists a series $1=H_0\leq H_1\leq\dots\leq H_n=H$ such that H_i is S-quasinormal in G and $|H_{i+1}:H_i|$=prime for all $0\leq i\leq n-1$. It is easily verified that if H and K are normal GSE subgroups of G, then HK is GSE in G. From this, every group G has a unique maximal generalized supersolvably embedded subgroup of G and it is denoted by GSE (G). The generalized hypercenter, genz_∞(G), is exactly the maximal generalized supersolvably embedded subgroup GSE (G) (see [3]). Furthermore, they proved the following:

(1) Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If $\Omega(P)\leq \text{genz}_\infty$(G), then G is p-nilpotent.

(2) A group G is supersolvable if and only if $\Omega(P)\leq \text{genz}_\infty$(G) for all Sylow subgroups P of G.

The present paper represents an attempt to extend and improve the above mentioned results.

2. Preliminaries

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([1; Theorem 2.9]) If K is a supersolvable subgroup of a group G, then genz_∞(G)K is supersolvable.

Lemma 2.2 ([2; Corollary 2]) Let P be a Sylow 2-subgroup of G. If P is quaternion-free and $\Omega_1(P) \leq Z$ (G), then G is 2-nilpotent.

Lemma 2.3 ([3; Corollary 3.6]) Let K $\lhd$ G. Then K $\cap$ genz $_\infty$ (G) $\leq$ genz $_\infty$ (K).

Lemma 2.4 ([5; Lemma 2.4]) If Q is a normal Sylow subgroup of a group G and K $\leq$ G, then genz $_\infty$ (K)Q/Q $\leq$ genz $_\infty$ (KQ/Q).

Lemma 2.5 ([6; VI, §9, Aufgabe 16]) If G is a minimal non-supersolvable group, then
(i) G has exactly one normal Sylow p-subgroup P for some prime p.
(ii) P/ $\Phi(P)$ is a minimal normal subgroup of G/ $\Phi(P)$.
(iii) The exponent of P is p at p>2 or at most 4 at p=2.

Lemma 2.6 Let P be a normal Sylow p-subgroup of G and assume that $\Omega(F(G)\cap P)K = \Omega(P)K$ is supersolvable, where K is a Hall p' -subgroup of G. Then G is supersolvable.

Proof: Suppose the result is false and let G be a counter example of minimal order. By [6; I, Satz 18.1], G/P $\cong$ K and so G is solvable. Then it is easy to show that the hypotheses
are inherited by all subgroups of G, so we may assume that G is a minimal non-supersolvable group. Then by Lemma 2.5(iii), P= $\Omega(P)$ and so G= $\Omega(P)K$. Thus G is supersolvable, a Contradiction.

3. Main Result

Theorem 3.1 Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If P is quaternion-free and $\Omega_1(F(G)\cap P) \leq$ genz $_\infty$ (G), then G is p-nilpotent.

Proof Suppose the result is false and let G be a counterexample of minimal order. Then G is not p-nilpotent and so G contains a minimal non-p-nilpotent subgroup, K, say. By [6; IV, Satz 5.4], K is a minimal non nil potent subgroup of G. By [6; III, Satz 5.2], |K|= $p^n q^m$ for a prime $p \neq q$, K has a normal Sylow p-subgroup K_p of exponent p at p>2 or at most 4 at p=2 and a non-normal cyclic Sylow q-subgroup K_q. Without loss of generality, we assume that $K_p \leq P$. Clearly $F(K) = K_p$. Then $\Omega_1(K_p) = \Omega_1(F(K)\cap K_p) \leq \Omega_1(F(G)\cap P) \leq$ genz $_\infty$ (G). By Lemma 2.1, genz $_\infty$ (G)K $_p$ is supersolvable. Since $\Omega_1(K_p)K_q \leq genz_\infty(G)K_q$, we have that $\Omega_1(K_p)K_q$ is supersolvable. Since p is the smallest prime divisor of |G|, it follows that $\Omega_1(K_p)K_q = \Omega_1(K_p)\times K_q$, that is, $K_q \leq C_G(\Omega_1(K_p))$. In fact, if $\Omega_1(K_p) = K_p$, then K_q is a normal subgroup of K, a contradiction. Thus $\Omega_1(K_p) \neq K_p$, and hence p=2. So, by [6; III, Satz 5.2], $K_2' = Z(K_2) = \Phi(K_2)$, K_2' is elementary abelian and K_2/K_2' is a chief factor of K. Then $\Omega_1(K_2) = K_2' \leq Z(K)$. Now, by applying Lemma 2.2, we conclude that K is 2-nilpotent; a final contradiction.
As a corollary of the proof of Theorem 3.1, we have:

Corollary 3.2 Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If $\Omega(F(G)\cap P) \leq$ genz $_\infty$ (G), then G is p-nilpotent.
Now we can prove:

Theorem 3.3 If $\Omega_1(F(G)\cap P) \leq$ genz $_\infty$ (G) for all Sylow subgroups P of G, then G is supersolvable or G has a section isomorphic to the quaternion group of order 8.

Proof If G has a section isomorphic to the quaternion group of order 8, then the result holds. Thus we can assume that G has no section isomorphic to the quaternion group of order 8. Theorem 3.1 implies that G is r-nilpotent, where r is the smallest prime dividing the order of G. Then G = RK, where R is a Sylow r-subgroup of G and K is a normal Hall r' -subgroup of G. Clearly $\Omega_1(F(K)\cap P) \leq \Omega_1(F(G)\cap P)$ for all Sylow subgroups P of K. By hypothesis and Lemma 2.3, $\Omega_1(F(K)\cap P) \leq$ genz $_\infty$ (G) $\cap$ K $\leq$ genz $_\infty$ (K) for all Sylow subgroups P of K. Then K is supersolvable by induction on the order of G. Hence K possesses an ordered Sylow tower and so Q is a normal Sylow q-subgroup of K, where q is the largest prime dividing the order of K. Since Q char K and K $\lhd$ G, we have that Q $\lhd$ G. By [6; I, Satz 18.1], G/Q $\cong$ L, where L is a Hall q' - subgroup of G. Clearly $\Omega_1(F(G/Q)\cap PQ/Q) \leq \Omega_1(F(G)Q/Q\cap PQ/Q)$ $= \Omega_1((F(G)\cap P)Q/Q) = \Omega_1(F(G)\cap P)Q/Q$ for all Sylow subgroups P of G, with (|P|, |Q|)=1. By hypothesis and Lemma 2.4, $\Omega_1(F(G/Q)\cap PQ/Q) \leq \Omega_1(F(G)\cap P)Q/Q \leq$ genz $_\infty$ (G)Q/Q $\leq$ genz $_\infty$ (G/Q). Then G/Q $\cong$ L is supersolvable by induction on |G|. Hence genz $_\infty$ (G)L is supersolvable by Lemma 2.1. Since $\Omega_1(F(G)\cap Q)$ char F (G) $\cap$ Q and

F(G) $\cap$ Q $\lhd$ G, we have that $\Omega_1(F(G)\cap Q)\lhd G$. By hypothesis $\Omega_1(F(G)\cap Q)L\leq$ genz$_\infty$(G)L$ and since genz$_\infty$(G)L is supersolvable, we have that $\Omega_1(F(G)\cap Q)L$ is supersolvable. Applying Lemma 2.6, we conclude that G is supersolvable.
The argument which established Theorem 3.3 can easily be adapted to yield the following three corollaries:

Corollary 3.4 If $\Omega(F(G)\cap P)\leq$ genz$_\infty$(G) for all Sylow subgroups P of G, then G is supersolvable.

Corollary 3.5 Assume that G is a group of odd order and that every subgroup of G of prime order is normal in G. Then G is supersolvable.

Corollary 3.6 If every subgroup of a group G of prime order or order 4 is S-quasinormal in G, then G is supersolvable.

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