A subgraph of conjugacy class graph of finite groups

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Abstract
Let $G$ be a finite group and let $G^{*}$ be the set of elements of prime power order of $G$. Let $\Gamma(G^{*})$ denote the prime graph built on the set of conjugacy class sizes of $G^{*}$. In this paper, we consider the situation when $\Gamma(G^{*})$ has some special vertices, and our aim is to investigate the influence of this property on the group structure of $G$.

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1. Introduction
Throughout the following, $G$ always denotes a finite group. For an element $g$ of a group $G$ we denote by $g^{*}$ the conjugacy class containing $g$. Let $cs(G)=\{||g^{*}|||g\in G\}$ be the set of the sizes of the conjugacy classes of $G$ and $V(G)=\{p$ prime $| p$ divides $n, n\in cs(G)\}$. In other words, $V(G)$ is the set of the primes dividing the size of some conjugacy class of $G$. The notation suggests that $V(G)$ is the set of vertices, a graph which we call $\Gamma(G^{*})$ the conjugacy class graph of $G$. The rest of our notation and terminology are standard.

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes. Many results are extensively studied by many authors (see, [1]-[7], [9]). For instance, a classical remark (see [9, Theorem 33.4]) concerning the influence of $cs(G)$ on the group structure of $G$ is the following:

Theorem A. Let $G$ be a group and $p$ a prime number. Then $p\not\in V(G)$ if and only if $G$ has a central Sylow $p$-subgroup.

In view of that, one can ask whether particular subsets of $cs(G)$ still encode nontrivial information on the structure of $G$. For instance, let $G^{*}$ be the set of elements of prime power orders of $G$. In this note, we study the interplay between the structure of a finite group $G$ and the set $cs(G^{*})$, a subset of $cs(G)$. We still use $V(G^{*})$ to denote the sets of vertices, a graph $\Gamma(G^{*})$ which we call a sub graph of conjugacy class graph of $G$ and obtain a complete extension of Theorem A. Our main result is the following:

Theorem B. Let $G$ be a group and $p$ a prime number. Then $p\not\in V(G^{*})$ if and only if $G$ has a Central Sylow $p$-subgroup.

2. Preliminaries
The following Lemma is one application of the Classification of the Finite Simple Groups, which is useful for our main results.

Lemma 2.1 ([8, Theorem 1]) Let $G$ be a transitive permutation group on a set $\Omega$ with $|\Omega|>1$. Then there exist a prime $p$ and an element $x\in G$ of order a power of $p$ such that $x$ acts without fixed points on $\Omega$. 
3. Proof of the Main Theorem

Proof of Theorem B. If a Sylow p-subgroup P of G is the center of G, then \( P \leq C_G(x) \) for all \( x \in G \) and \( p \not\in \mathbb{V}(G) \).

Conversely, suppose that \( p \) is a prime such that \( p \not\in \mathbb{V}(G) \) and let \( P \in \text{Syl}_p(G) \), we prove that \( P \leq Z(G) \). At first we conclude that P is a unique Sylow p-subgroup of G, that is, \( P \lhd G \). Let \( \Omega = \text{Syl}_p(G) \). If \( |\Omega|=1 \), we are done. Assume that \( |\Omega|>1 \). We consider the conjugacy action of G on \( \Omega \). By Sylow Theorems we know that G acts transitively on \( \Omega \). Thus by Lemma 2.1, there exists a prime \( r \) and an \( r \)-element \( g \in G \) such that \( g \) acts without fixed point on \( \Omega \), that is, for any \( P \in \text{Syl}_r(G) \), \( P \neq P^g \). Suppose \( r \neq p \).

Then \( p \) does not divide \( |g^G| = |G: C_G(g)| \) according to the previous argument. So there exists an element \( w \in G \) such that \( P^w \leq C_G(g) \) by Sylow Theorems, which implies that \( (P^w)^f = P^w \), a contradiction. If \( r=p \), also by Sylow Theorems, there exists an element \( z \in G \) such that \( g \in P^z \), which implies that, \( P^z = P^g \) again a contradiction. Now by Schur-Zassenhaus Theorem, there exists a Hall \( p^r \)-subgroup K of G such that \( G=PK \) and all Hall \( p^r \)-subgroup of G are conjugate. Let \( x \) be an element of prime power order of K. Then \( p \) does not divide \( |x^K| \) by the previous argument. Hence \( P \leq C_G(x) \). Since K can be generated by elements of prime power orders, we have that \( P \leq C_G(x) \). So \( G=P \times K \).

In the following we only need to prove P is abelian. For any element \( y \in P \), then \( p \) does not divide \( |y^K| = |G: C_G(y)| \) according to the hypotheses. Thus \( P \leq C_G(x) \) and P is abelian. Thus G has a central Sylow p-subgroup.

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5. References

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