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A subgraph of conjugacy class graph of finite groups

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Abstract
Let G be a finite group and let G^* be the set of elements of prime power order of G. Let $\Gamma(G^*)$ denote the prime graph built on the set of conjugacy class sizes of G^*. In this paper, we consider the situation when $\Gamma(G^*)$ has some special vertices, and our aim is to investigate the influence of this property on the group structure of G.

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1. Introduction

Throughout the following, G always denotes a finite group. For an element g of a group G we denote by g^G the conjugacy class containing g. Let cs (G)=$\{|g^G| \mid g\in G\}$ be the set of the sizes of the conjugacy classes of G and V(G)=$\{p \text{ prime} \mid p \text{ divides } n, n\in cs(G)\}$. In other words, V (G) is the set of the primes dividing the size of some conjugacy class of G. The notation suggests that V (G) is the set of vertices, a graph which we call $\Gamma(G^*)$ the conjugacy class graph of G. The rest of our notation and terminology are standard.

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes. Many results are extensively studied by many authors (see, [1]-[7], [9]). For instance, a classical remark (see [9, Theorem 33.4]) concerning the influence of cs (G) on the group structure of G is the following:

Theorem A. Let G be a group and p a prime number. Then $p\notin V(G)$ if and only if G has a central Sylow p-subgroup.

In view of that, one can ask whether particular subsets of cs (G) still encode nontrivial information on the structure of G. For instance, let G^* be the set of elements of prime power orders of G. In this note, we study the interplay between the structure of a finite group G and the set cs (G^*), a subset of cs (G). We still use V (G^*) to denote the sets of vertices, a graph $\Gamma(G^*)$ which we call a sub graph of conjugacy class graph of G and obtain a complete extension of Theorem A. Our main result is the following:

Theorem B. Let G be a group and p a prime number. Then $p\notin V(G^*)$ if and only if G has a Central Sylow p-subgroup.

2. Preliminaries

The following Lemma is one application of the Classification of the Finite Simple Groups, which is useful for our main results.

Lemma 2.1 ([8, Theorem 1]) Let G be a transitive permutation group on a set Ω with $|\Omega|>1$.Then there exist a prime p and an element $x\in G$ of order a power of p such that x acts without fixed points on Ω.

3. Proof of the Main Theorem

Proof of Theorem B. If a Sylow p-subgroup P of G is the center of G, then P $\leq C_G(x)$ for all x $\in$ G and p $\notin$ V(G^*).

Conversely, suppose that p is a prime such that p $\notin$ V(G^*) and let P $\in Syl_p(G)$, we prove that P $\leq$ Z(G). At first we conclude that P is a unique Sylow p-subgroup of G, that is, P $\lhd$ G. Let $\Omega = Syl_p(G)$. If $|\Omega|$=1, we are done. Assume that $|\Omega|$>1. We consider the conjugacy action of G on Ω. By Sylow Theorems we know that G acts transitively on Ω. Thus by Lemma 2.1, there exists a prime r and an r-element g $\in$ G such that g acts without fixed point on Ω, that is, for any P $\in Syl_p(G)$, $P^g \neq P$. Suppose r $\neq$ p. Then p does not divide $|g^G|$=|G:$C_G(g)|$ according to the previous argument. So there exists an element w $\in$ G such that $P^w \leq C_G(g)$ by Sylow Theorems, which implies that $(P^w)^g = P^w$, a contradiction. If r=p, also by Sylow Theorems, there exists an element z $\in$ G such that $g \in P^z$, which implies that, $P^{zg} = P^g$ again a contradiction. Now by Schur-Zassenhaus Theorem, there exists a Hall p'-subgroup K of G such that G=PK and all Hall p'-subgroup of G are conjugate. Let x be an element of prime power order of K. Then p does not divide $|x^G|$ by the previous argument. Hence $P \leq C_G(x)$. Since K can be generated by elements of prime power orders, we have that $P \leq C_G(x)$. So G=P $\times$ K.

In the following we only need to prove P is abelain. For any element y $\in$ P, then p does not divide $|y^G| = |G : C_G(y)|$ according to the hypotheses. Thus $P \leq C_G(x)$ and P is abelian. Thus G has a central Sylow p-subgroup.

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