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Common fixed point theorem of two mappings in menger PM: Spaces

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Abstract
By using the concept of contraction of the two mappings in menger PM-spaces, the paper introduces coincidence and common fixed points in menger PM- spaces. The paper extends the result of the paper "Marwan Amin Kutbi" [5]

Keywords: point theorem, mappings, PM-spaces, coincidence

1. Introduction
The concept of Probabilistic metric space (PM- Space) was first studied by Menger [1] in 1942 See also [2-4] There after some fixed point results were given by Schgal and Bharucha-Reid [6, 7] By using contractive condition in probabilistic metric space, they proved a unique fixed point result which was an extension of Banach's work [8] regarding fixed point theorem in metric space. Many fixed point result were proved in the space; see [13-20] In particular, Dutta *et al.* [21] Nonlinear Ψ – contractive mapping in Menger PM-space and proved the result using this contractive mapping in G-complete Menger PM-Spaces. Weaking this Ψ – contractive mapping, Marian Amin Kutbi, Dhananjag Gopal, Calogera Vetro and Wutiphol Sintumaverat [5] gave some fixed point results in G-complete and M-complete Menger PM-spaces.
On the basis of studying the result [5]; we defined Ψ – contraction of one mapping with respect to f and proved the coincidence and fixed points of the two mappings in the Menger PM-space.
Here, we state some definitions which are needed to prove our result. We denote the set of real numbers by R, byR^+, the set of non-negative real numbers and by N, the set of positive integers.

Definition 1.1 [9, 22] A mapping $f: R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $inf_{t\varepsilon R} F(t) = 0$ and$sup_{t\varepsilon R}\ F(t) = 1$
We denote by D^+ the set of all distribution functions, while $H\varepsilon D^+$ will always be denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$$

Definition 1.2: [22] A binary operation $T: [0,1] \times [0,1] \to [0,1]$ is a continuous $t - norm$ if the following conditions hold:
(a) T is commutative and associative,
(b) T is continuous.
(c) T (a, 1) = a for all $a\varepsilon[0,1]$

(d) T(a, b), whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$

The following are three basic continuous $t - norm$ from the literature :

i. The minimum $t - norm$, say $T_{M,}$ defined by $T_M(a, b) = min(a, b)$.

ii. The product $t - norm$, say $T_{p,}$ defined by $T_p(a, b) = a.b$.

iii. The Lukasiewicz $t - norm\, T_{L,}$ defined by $T_L(a, b) = max(a, b)$

These $t - norm$ are related in the following way : $T_L \leq T_p \leq T_M$

Definition 1.3: [5] A menger PM space is a triple (X , F, T) where X is a nonempty set, T is a continuous $t - norm$ and f is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denoted the value of F at the pair x, y, the following conditions hold

$F_{x,y}(t) = H(t)$ if and only if $x = y$ for all $t\varepsilon R^+$,

$F_{x,y}(t) = F_{x,y}(t)$ for all, $x, y, z\varepsilon R^+$,

$F_{x,y}(t + s) \geq T\left(F_{x,z}(t), F_{z,y}(t)\right)$ for all $x, y, z\varepsilon R^+$.

Definition 1.4: [5] Let (X , F, T) be a Menger PM-space, then

(i) A sequence (x_n) in X is said to be convergent to $x\varepsilon X$ if, every $\epsilon > 0$ and, $\lambda > 0$ there exists a positive integer N such that $F_{x,y}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(ii) A sequence (x_n) in X is called Cauchy sequence if, for every $\epsilon > 0$ and, $\lambda > 0$ there exists a positive integer N such that $F_{x_n, y_n}(\epsilon) > 1 - \lambda$ whenever $m, n \geq N$.

(iii) A Menger PM-space is said to be M-complete if every Cauchy sequence in X is convergent to a point in X.

(iv) A sequence (x_n) is called G-Cauchy if $lim_{n \to \infty} F_{x_n y_n}(t) = 1$ for each $m\varepsilon N$ and $t > 0$.

(v) The space (X , F, T) is called G-complete if every G-Cauchy sequence in X is convergent.

The following class of functions was introduced in [10] and will be used in proving our results in the next section.

Definition 1.5: [11] A function $\psi: R^+ \to R^+$ is said to be $\psi -$ function if it satisfies the following conditions:

(i) $\psi(t) = 0$ if and only if $t = 0$.

(ii) $\psi(t)$ is strictly increasing and $\psi(t) \to \infty$ as $t \to \infty$

(iii) ψ is left continuous in $(0, \infty)$

(iv) ψ is continuous at 0

Definition 1.6: Let (X, F, T) be a Manger PM- space and T, $f: X \times X$ be the self mappings. A point x in X is called a coincidence point (common fixed point) if $Tx = fx$. Also the pair T, $f: X \times X$ of mappings are weakly compatible if they commute on the set of coincidence points.

Definition 1.7: (23) Let (X , F, T) be a Manager PM-space. The probabilistic metric F is triangular if it satisfies the condition

$$\frac{1}{F_{x,y}(t)} - 1 \leq \left(\frac{1}{F_{x,z}(t)} - 1\right) + \left(\frac{1}{F_{z,y}(t)} - 1\right)$$

For every $x, y, z \in X$ and each $t > 0$

In the sequel, the class of all φ– functions will be denoted by Φ. Also we denoted ψ the class of all continuous non-decreasing functions such that $\psi(0) = 0$ and $\psi^n(a_n) \to 0$ as $n \to \infty$

Theorem 1.1 [5] Let (X, F, T) be a G-complete Menger space and $f : X \to X$ be a mapping satisfying the following inequality:

$$\frac{1}{F_{Tx,y}(\varphi(t))} - 1 \leq \psi\left(\frac{1}{F_{x,y}(\varphi(t))} - 1\right) \qquad (1.1)$$

Where $x, y \in X$, $c \in (0,1)$, $\varphi \in \psi$ and $t > 0$ such that $F_{x,y}(\varphi(t)) > 0$ then f has a unique fixed point.

A mapping $f: X \to X$ satisfying condition (1.1) is usually called $\psi -$ contractive mapping. However for some discussion on this notion and theorem 1.1 the reader can refer to the recent paper of Gopal *et al* [23] where analogous result are proved by using some control function

2. The Main Results

Theorem 2.1 Let (X, F, T) be a menger space and $T : f : X \to X$ be the mapping satisfying the following inequality

$$\frac{1}{F_{Tx,Ty}(\varphi(ct))} - 1 \le \psi\left(\frac{1}{F_{fx,fy}(\varphi(t))} - 1\right) \quad (2.1)$$

Where $x, y \in X, c \in (0,1), \varphi \varepsilon \Phi, \psi \varepsilon \Psi$ and $t > 0$ such that $F_{Tx,Ty}(\varphi(ct))$. If the range of $T(X) \subset f(X)$ is a G-complete subspace of, then f and T have coincidence point.

Further if the pair of mapping (T, f) is weakly compatible, then f and t have a common fixed point.

Proof Let $x_0 \varepsilon X$. Take a point $x_1$in X such that $T(x_0) = f(x_1)$. This is possible as the range of f contain the range of T. Continuing in this way, for every x_n in X. One can find a x_{n+1} such that $y_n = Tx_n = fx_{n+1}$ Without loss of generality assume the $y_{n+1} \ne y_n$ for all $n \varepsilon N$; otherwise f and T have a coincidence point and there is nothing to prove. In case $y_{n+1} \ne y_n$

$$\frac{1}{F_{y_1,y_2}(\varphi(t))} - 1 = \frac{1}{F_{Tx_1,Tx_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1$$

$$\le \psi\left(\frac{1}{F_{Tx_1,Tx_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1\right)$$

$$= \psi\left(\frac{1}{F_{Tx_1,Tx_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1\right) \quad (2.2)$$

From (2.2) we deduce that $F_{y_1,y_2}(\varphi(t)) > 0$ and $F_{y_1,y_2}\left(\varphi\left(\frac{t}{c}\right)\right) > 0$. Again by applying (2.1), we get

$$\frac{1}{F_{y_2,y_3}(\varphi(t))} - 1 = \frac{1}{F_{Tx_2,Tx_3}(\varphi(t))} - 1$$

$$\le \psi\left(\frac{1}{F_{fx_1,fx_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1\right)$$

$$= \psi\left(\frac{1}{F_{y_1,y_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1\right) \quad (2.2)$$

that is

$$\frac{1}{F_{y_2,y_3}(\varphi(t))} - 1 \le \psi\left(\frac{1}{F_{y_1,y_2}\left(\varphi\left(\frac{t}{c}\right)\right)} - 1\right) \quad (2.3)$$

On using (2.2) and the hypothesis that ψ is non – decresing the above inequality (2.3) becomes

$$\frac{1}{F_{y_2,y_3}(\varphi(t))} - 1 \le \psi^2\left(\frac{1}{F_{y_0,y_1}\left(\varphi\left(\frac{t}{c^2}\right)\right)} - 1\right) \quad (2.4)$$

Repeating the above procedure successively n times, we obtain

$$\frac{1}{F_{y_n,y_{n+1}}(\varphi(t))} - 1 \le \psi^n\left(\frac{1}{F_{y_0,y_1}\left(\varphi\left(\frac{t}{c^n}\right)\right)} - 1\right)$$

If we change y_0 with y_r in the previous inequality then for all $n > r$, We get

$$\frac{1}{F_{y_n,y_{n+1}}(\varphi(c^r t))} - 1 \leq \psi^{n-r}\left(\frac{1}{F_{y_0,y_1}\left(\varphi\left(\frac{c^r t}{c^{n-r}}\right)\right)} - 1\right) \qquad (2.5)$$

Since $\psi^n(a_n) \to 0$ whenever $a_n \to 0$therefore the above inequality implies that

$$lim_{n\to\infty} F_{y_n,y_{n+1}}(\varphi(c^r t)) = 1 \qquad (2.6)$$

Now let $\epsilon > 0$ be given, then by using the properties (i) to (iv) of a function we can find $r\varepsilon N$ such that $\varphi(c^r t) < \epsilon$. It follows from (2.6) that

$$lim_{n\to\infty} F_{y_n,y_{n+1}}(\varphi(\epsilon)) = 1 \quad \text{for every } \epsilon > 0 \ (2.7)$$

Hence y_n is a G-Cauchy sequence in $f(X)$and $f(X)$ is G-complete. Therefore $y_n \to y$ as $n \to \infty$, for some $y\varepsilon\, f(X)$. Consequently we obtain a point u in X such that $f(u) = v$. Now we show that u is coincidence point of $f$and T.

Since,

$$F_{Tu,u}(\epsilon) \geq T\left(F_{Tu,y_{n+1}}\left(\frac{\epsilon}{2}\right), F_{Ty_{n+1},u}\left(\frac{\epsilon}{2}\right)\right) (2.8)$$

By using the properties (i) and (iv) of a function φ, we can find $s > 0$ such that $\varphi(s) < \frac{\epsilon}{2}$. Again since $y_n \to y$ as $n \to \infty$, then there exists $n\varepsilon N$ such that, for all $n > n_0$ we have ……$F_{y_n,u}(\varphi(s)) > 0$.

Therefore, for every $n > n_0$ we obtain

$$\frac{1}{F_{y_{n+1},Tu}\left(\varphi\left(\frac{\epsilon}{2}\right)\right)} - 1 \leq \frac{1}{F_{Tx_{n+1},Tu}(\varphi(s))} - 1$$

$$\leq \psi\left(\frac{1}{F_{fx_{n+1},fu}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right)$$

$$= \psi\left(\frac{1}{F_{y_n,fu}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right)$$

….

Since Ψ is continuous at 0 and $\Psi(0) = 0$, we obtain

$$lim_{n\to\infty} F_{y_{n+1},T(u)}\left(\frac{\epsilon}{2}\right) = 1$$

From (2.8) and (2.9), we get $F_{fv,Tw}(\epsilon) = 1$ for every $\epsilon > 0$,which implies that $Tu = fu$.

This proves that u is coincidence point of $f$and T. Further suppose $Tu = fu = v$. Since $f$and T are compatible, then $fTu = fTv$ Obviously, $Tv = fv$. Now we show that $fv = v$.

$$\frac{1}{F_{y_{n+1},fv}\left(\varphi\left(\frac{\epsilon}{2}\right)\right)} - 1 = \frac{1}{F_{Tx_{n+1},Tv}\left(\varphi\left(\frac{\epsilon}{2}\right)\right)} - 1$$

$$\leq \frac{1}{F_{Tx_{n+1},Tw}(\varphi(s))} - 1$$

$$\leq \psi\left(\frac{1}{F_{fx_{n+1},fv}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right)$$

$$= \psi\left(\frac{1}{F_{y_n,u}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right) \qquad (2.10)$$

Since Ψ is continuous at 0 and $\Psi(0) = 0$, we obtain

$$lim_{n\to\infty} F_{y_{n+1},f(v)}\left(\frac{\epsilon}{2}\right) = 1$$

From (2.8) and (2.10), we get $F_{v,fw}(\varepsilon) = 1$ for every $\epsilon > 0$, which implies that $v = w$ Hence v is fixed point of T and f. Further suppose that w is another fixed pint of T and f, then

$$\frac{1}{F_{y_{n+1},w}\left(\varphi\left(\frac{\epsilon}{2}\right)\right)} - 1 = \frac{1}{F_{Tx_{n+1},Tw}\left(\varphi\left(\frac{\epsilon}{2}\right)\right)} - 1$$

$$\leq \frac{1}{F_{Tx_{n+1},Tw}(\varphi(s))} - 1$$

$$\leq \psi\left(\frac{1}{F_{fx_{n+1},fw}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right)$$

$$= \psi\left(\frac{1}{F_{y_n,w}\left(\varphi\left(\frac{s}{c}\right)\right)} - 1\right)$$

Since Ψ is continuous at 0 and $\Psi(0) = 0$, we obtain

$$lim_{n\to\infty} F_{y_{n+1},w}\left(\frac{\epsilon}{2}\right) = 1$$

From (2.8) and (2.10), we get $F_{v,w}(\epsilon)$=1 for every $\epsilon > 0,$ which implies that $v = w$ Thus we have guarantee of uniqueness of fixed point.

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