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An application of generalized Tsalli's-Havrda-Charvat entropy in coding theory through a generalization of Kraft inequality

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Abstract
A parametric mean length is defined as the quantity

$$_{\alpha\beta}L_u = \frac{1}{\alpha-1}\left[1-\frac{\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}}\right],$$

where $\alpha > 0\ (\neq 1),\ \beta > 0, u_i > 0, D > 1$ is an integer, $\sum p_i = 1$. This being the useful mean length of code words weighted by utilities, u_i. Lower and Upper bounds for $_{\alpha\beta}L_u$ are derived in terms of 'useful' Tsalli's-Havrda-Charvat information measure for power probability distribution p_i^{β}.

Keywords: Tsalli's Entropy, Useful Tsalli's entropy, Utilities, Kraft inequality, Holder's inequality
AMS Subject classification: 94A15, 94A17, 94A24, 26D15.

1. Introduction

Consider the following model for a random experiment S,

$S_N = [E;\ P;\ U]$,

where $E = (E_1, E_2, ..., E_N)$ is a finite system of events happening with respective probabilities $P = (p_1, p_2, ..., p_N), p_i \geq 0, \sum p_i = 1$ and credited with utilities $U = (u_1, u_2, ..., u_N),\ u_i > 0,\ i = 1, 2, ..., N$. Denote the model by S_N, where,

$$S_N = \begin{bmatrix} E_1, E_2, ..., E_N \\ p_1, p_2, ..., p_N \\ u_1, u_2, ..., u_N \end{bmatrix}. \quad (1.1)$$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [2] proposed a measure of information called 'useful information' for this scheme, given by

$$H(U;\ P) = -\sum u_i p_i \log(p_i), \quad (1.2)$$

where $H(U;P)$ reduces to Shannon's [15] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each *i*. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D\ (D > 1)$.

Guiasu and Picard [4] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, ..., w_N$ having lengths $n_1, n_2, ..., n_N$ and satisfying Kraft's inequality [3].

$$\sum_{i=1}^{N} D^{-n_i} \le 1. \qquad (1.3)$$

Where D is the size of the code alphabet. The useful mean length L_u of code was defined as:

$$L_u = \frac{\sum u_i\, n_i\, p_i}{\sum u_i\, p_i}, \qquad (1.4)$$

and the authors obtained bounds for it in terms of $H(U;\, P)$. Generalized coding theorems by considering different generalized measures under condition (1.3) of unique decipherability were investigated by several authors, see for instance the papers [4, 8-10, 13]. In this paper, we study some coding theorems by considering a new function depending on the parameters α, β and a utility function. Our motivation for studying this new function is that it generalizes 'useful' information measure already existing in the literature such Tsalli's entropy [17], Havrda-Charvat [6] etc.

2. Coding Theorems

In this section, we define a new information measure as :

$${}_{\alpha\beta}H(U;\, P) = \frac{1}{\alpha - 1}\left[1 - \frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right], \qquad (2.1)$$

where $\beta > 0, \alpha > 0\ (\neq 1)$, $u_i > 0$, $p_i \ge 0$, $i = 1, 2, \ldots, N$ and $\sum p_i = 1$.

(i) If $\beta = 1$, Then (2.1) becomes a 'useful' information measure

i.e., $${}_{\alpha}H(U;\, P) = \frac{1}{\alpha - 1}\left[1 - \frac{\sum u_i p_i^{\alpha}}{\sum u_i p_i}\right]. \qquad (2.2)$$

(ii) When $u_i = 1$ for each *i*, i.e., when the utility aspect is ignored, $\sum p_i = 1$, and $\beta = 1$, then (2.1) reduces to Tsalli's-Havrda-Charvat entropy.

i.e., $${}_{\alpha}H(P) = \frac{1}{\alpha - 1}\left[1 - \sum p_i^{\alpha}\right]. \qquad (2.3)$$

(iii) When $\alpha \to 1$, and $\beta = 1$, then (2.1) reduces to a measure of 'useful' information due to Hooda and Bhaker [1].

i.e., $$H(U;P) = -\frac{\sum u_i p_i \log(p_i)}{\sum u_i p_i}. \qquad (2.4)$$

(iv) When $u_i = 1$ for each *i*, then (2.1) reduced to Satish and Arun [13] entropy.

i.e., $${}_{\alpha\beta}H(U;\, P) = \frac{1}{\alpha - 1}\left[1 - \frac{\sum p_i^{\alpha\beta}}{\sum p_i^{\beta}}\right]. \qquad (2.5)$$

(v) When $u_i = 1$ for each *i*, i.e., When the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$, and $\alpha \to 1$, the measure (2.1) reduces to Shannon's entropy [15].

i.e., $$H(P) = -\sum p_i \log(p_i). \qquad (2.6)$$

Further consider,

Definition: The 'useful' mean length ${}_{\alpha\beta}L_u$ with respect to 'useful' R-norm information measure is defined as :

$${}_{\alpha\beta}L_u = \frac{1}{\alpha - 1}\left[1 - \frac{\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}}\right], \qquad (2.7)$$

under the condition, $$\sum u_i D^{-n_i \alpha} \le \sum u_i p_i^{\alpha\beta}. \qquad (2.8)$$

Clearly the inequality (2.8) is the generalization of Kraft's inequality (1.3). A code satisfying (2.8) would be termed as a 'useful' personal probability code. D (D>2) is the size of the code alphabet. When, $u_i = 1$ for each i and $\beta = 1, \alpha = 1$, (2.8) reduces to (1.3).

(i) For $u_i = 1$ for each i and $\beta = 1$, and $\alpha \to 1$, ${}_{\alpha}L_u$ becomes the optimal code length defined by Shannon [15].

(ii) For $u_i = 1$ for each i and $\beta = 1$, then (2.7) becomes a new mean code word length corresponding to the Tsalli's entropy.

i.e., $${}_{\alpha}L = \frac{1}{\alpha - 1}\left[1 - \sum p_i\, D^{-n_i(\alpha-1)}\right]. \tag{2.9}$$

(iii) If $\beta = 1$, then (2.7) becomes a new mean codewords length corresponding to the entropy (2.2).

i.e., $${}_{\alpha}L_u = \frac{1}{\alpha - 1}\left[1 - \frac{\sum u_i p_i\, D^{-n_i(\alpha-1)}}{\sum u_i p_i}\right].$$

(iv) If $u_i = 1$, then (2.7) becomes a mean codewords length corresponding to the entropy (2.5).

i.e., $${}_{\alpha\beta}L = \frac{1}{\alpha - 1}\left[1 - \frac{\sum p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum p_i^{\beta}}\right].$$

We establish a result, that in a sense, provides a characterization of ${}_{\alpha\beta}H\,(U\,;\,P)$ under the condition of unique decipherability.

Theorem 2.1. Let $u_i, p_i, n_i,\ i = 1, 2, ..., N$, satisfy the inequality (2.8). Then

$${}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H\,(U\,;P),\quad 1 \neq \alpha > 0, \beta > 0. \tag{2.10}$$

Proof: By Holder's inequality, we have

$$\left(\sum_{i=1}^{N} x_i^{p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} y_i^{q}\right)^{\frac{1}{q}} \leq \sum_{i=1}^{N} x_i y_i\,, \tag{2.11}$$

where $p^{-1} + q^{-1} = 1; p(\neq 0) < 1, q < 0$ or $q(\neq 0) < 1, p < 0; x_i, y_i > 0$ for each i.

Setting, $p = \dfrac{(\alpha - 1)}{\alpha}, q = 1 - \alpha$, and

$$x_i = \left(\frac{u_i p_i^{\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{\alpha}{\alpha-1}} D^{-n_i\alpha},\quad y_i = \left(\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{1-\alpha}}, \tag{2.12}$$

Putting these values in (2.11) and using the inequality (2.8), we get

$$\left(\frac{\sum u_i p_i^{\beta}\ D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha-1}} \leq \frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}. \tag{2.13}$$

It implies

$$\left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{\alpha}{1-\alpha}} \leq \left(\frac{\sum u_i p_i^{\beta}\ D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}}\right)^{\frac{\alpha}{\alpha-1}}. \tag{2.14}$$

Now consider two cases:

Case 1: Let $0 < \alpha < 1$. Raising both sides of (2.14) to the power $(\alpha - 1)/\alpha$, we get

$$\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}} \leq \frac{\sum u_i p_i^{\beta}\ D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}} \tag{2.15}$$

Since, $1/(\alpha - 1) < 0$ for $0 < \alpha < 1$, we get from (2.15) the inequality (2.10).

Case 2: Let $\alpha > 1$. The proof follows on the same lines.

It is clear that the equality in (2.10) is true if and only if

$$D^{-n_i} = p_i^{\beta}$$

which implies that

$$n_i = \log_D \frac{1}{p_i^{\beta}} \tag{2.16}$$

Thus, it is always possible to have a codeword satisfying the requirement

$$\log_D \frac{1}{p_i^{\beta}} \le n_i < \log_D \frac{1}{p_i^{\beta}} + 1,$$

which is equivalent to

$$\frac{1}{p_i^{\beta}} \le D^{n_i} < \frac{D}{p_i^{\beta}} \,. \tag{2.17}$$

In the following theorem, we give an upper bound for ${}_{\alpha\beta}L_u$ in terms of ${}_{\alpha\beta}H(U;P)$.

Theorem 2.2. By properly choosing the lengths $n_1, n_2, ..., n_N$ in the code of Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisy the following inequality:

$${}_{\alpha\beta}L_u < D^{(1-\alpha)}\left[{}_{\alpha\beta}H(U;P)\right] + \frac{1}{\alpha-1}\left(1 - D^{(1-\alpha)}\right) \tag{2.18}$$

Proof: From (2.17), it is clear that

$$D^{-n_i} > D^{-1}\, p_i^{\beta}\,. \tag{2.19}$$

We have again the following two possibilities.

(i) Let $\alpha > 1$. Raising both sides of (2.19) to the power $(\alpha - 1)$, we have

$$D^{-n_i(\alpha-1)} > D^{1-\alpha}\, p_i^{\beta(\alpha-1)}\,.$$

Multiplying both sides by $u_i p_i^{\beta}$ and then summing over i. we get

$$\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)} > D^{(1-\alpha)} \sum u_i p_i^{\alpha\beta}\,. \tag{2.20}$$

Obviously (2.20) can be written as

$$\frac{\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}} > D^{(1-\alpha)} \frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}} \tag{2.21}$$

Since $\alpha - 1 > 0$ for $\alpha > 1$, we get the inequality (2.18) from (2.21).

(ii) If $0 < \alpha < 1$, the proof follows similarly. But the inequality (2.21) is reversed.

Theorem 2.3. For arbitrary $N \in \mathbb{N}, 1 \ne \alpha > 0, \beta > 0$, and for every codeword lengths $n_i, i = 1, 2, ..., N$ of Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisy the following inequality:

$${}_{\alpha\beta}L_u \ge {}_{\alpha\beta}H(U;P) > D\left[{}_{\alpha\beta}H(U;P)\right] + \frac{1}{\alpha-1}(1-D)\cdot \tag{2.22}$$

Proof: Suppose,

$$\bar{n}_i = \log_D \frac{1}{p_i^{\beta}}\,,\ \beta > 0\,. \tag{2.23}$$

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy the equality in Holder's inequality (2.11). Moreover, $\bar{n}_i$ satisfies (2.8). Suppose $\bar{n}_i$ is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, then obviously, $\bar{n}_i$ satisfies (2.8).

Since $1 \ne \alpha > 0$, $\beta > 0$, we have

$$\frac{\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}} \le \frac{\sum u_i p_i^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum u_i p_i^{\beta}}$$

$$< D\left(\frac{\sum u_i p_i^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum u_i p_i^{\beta}}\right). \tag{2.24}$$

Since, $\frac{\sum u_i p_i^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum u_i p_i^{\beta}} = \frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}$.

Hence (2.24) becomes

$$\frac{\sum u_i p_i^{\beta} D^{-n_i(\alpha-1)}}{\sum u_i p_i^{\beta}} \leq \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}} \right) < D\left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}} \right).$$

Which gives (2.22).

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