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Ulam stability of radical functional equation in the sense of Hyers, Rassias and Gavruta

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Abstract
The aim of this paper is to obtain the general solution of a radical functional equation of the form

$$s\left(x+y+2\sqrt{xy}\right)=s(x)+s(y)$$

and investigate its generalized Hyers-Ulam-Rassias stability. Further, we extend the results pertinent to D.H. Hyers, Th.M. Rassias and J.M. Rassias. We also provide counter-examples for critical values relevant to Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by mixed product-sum of powers of norms.

Keywords: Radical functional equation, Generalized Hyers-Ulam stability.
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1. Introduction

The stability of functional equations is an interesting topic that has been dealt for the last six decades. In mathematics, a stipulation in which a slight disturbance in a system does not create a considerable disturbing consequence on that system. An equation is said to be stable if a slightly different solution is close to the exact solution of that equation. In mathematical models of physical problems, the deviations in measurements will result with errors and these deviations can be dealt with the stability of equations. Hence the stability of equations is essential in mathematical models. A stable solution will be sensible in spite of such deviations.

An interesting and eminent talk given by S.M. Ulam [28] in 1940, inspired to study the investigation of stability of functional equations. He posed the following question pertaining to the stability of homomorphisms in groups:

Let $\mathcal{X}$ be a group and $\mathcal{Y}$ be a metric group with metric $d(\cdot,\cdot)$. Given $\epsilon \geq 0$ does there exist a $\delta \geq 0$ such that if a function $f:\mathcal{X}\rightarrow\mathcal{Y}$ fulfills $d(f(xy), f(x)f(y) \leq \delta$ for all $x,y\in\mathcal{X}$, then there exists a homomorphism $g:\mathcal{X}\rightarrow\mathcal{Y}$ with $d(f(x),g(x))\leq\epsilon$ for all $x\in\mathcal{X}$?

If the answer is confirmative, then the functional equation for homomorphism is said to be stable. The foremost answer to the question of S.M. Ulam was provided by D.H. Hyers [12]. He brilliantly answered the question of Ulam by considering $\mathcal{X}$ and $\mathcal{Y}$ as Banach spaces. The result of Hyers is stated in the following celebrated theorem. The stability resulted provided by Hyers in the following theorem is referred as Hyers-Ulam stability.

Theorem 1.1. (D.H. Hyers [12])*Let $f:\mathcal{X}\rightarrow\mathcal{Y}$ be a mapping between Banach spaces such that*

$$\|f(x+y)-f(x)-f(y)\|\leq\delta \tag{1.1}$$

for all $x,y\in\mathcal{X}$ and for some $\delta>0$, then the limit

$$A(x)=\lim_{n\rightarrow\infty}2^{-n}f(2^{-n}x) \tag{1.2}$$

exists for each $x\in\mathcal{X}$. Then there exist a unique additive mapping $A:\mathcal{X}\rightarrow\mathcal{Y}$ such that

$$\|f(x)-A(x)\|\leq\delta \tag{1.3}$$

for all $x\in\mathcal{X}$. In addition if $f(tx)$ is continuous in t for each fixed $x\in\mathcal{X}$, then A is linear.

After Hyers gave a positive answer to Ulam's question, a huge number of papers have been published in association with various simplifications of Ulam's problem and Hyers theorem. Hyers theorem was generalized by T. Aoki [1] in 1950 for additive mappings.

Since there is no elucidation for the boundedness of Cauchy difference $f(x+y)-f(x)-f(y)$in the expression of (1.1), in the year 1978, Th.M. Rassias [26] tried to weaken the stipulation for the Cauchy difference and thrived in proving what is now known to be the Hyers-Ulam-Rassias stability for the Additive Cauchy Equation. This jargon is reasonable because the theorem of Th.M. Rassias has strongly persuaded many mathematicians studying stability problems of functional equation.

During 1982-1989, J.M. Rassias ([18-20]) provided a further generalization of the result of D.H. Hyers and established a theorem using weaker conditions controlled by a product of different powers of norms and this type of stability is termed as Ulam-Gavruta-Rassias stability.

A generalized and modified form of the theorem evolved by Th.M. Rassias was promoted by P. Gavruta [10] who replaced the unbounded Cauchy difference by motivating into study a general control function within the viable approach designed by Th.M. Rassias and this type of stability is celebrated as Generalized Hyers-Ulam stability.

The investigation of stability of functional equations involving with the mixed type product-sum of powers of norms is introduced by K. Ravi et al.[23] is called J.M. Rassias stability controlled by mixed product-sum of powers of norms.

For the last four decades, many mathematicians have investigated various stability results of several functional equations like additive, quadratic, cubic, quartic, quintic, sextic, septic, octic and nonic, functional equations (refer [2-4, 7, 8, 14, 16, 17, 21, 22, 29, 30]). There are many interesting results concerning this problem and many research monographs are also available in functional equations, one can see ([5, 6, 13, 15, 27]).

K. Ravi and P. Narasimman [24] obtained the generalized Hyers-Ulam stability of a radical reciprocal-quadratic functional equation

$$f\left(\sqrt{x^2+y^2}\right)=\frac{f(x)f(y)}{f(x)+f(y)} \tag{1.4}$$

in Banach spaces. The reciprocal-quadratic function $f(x)=\frac{c}{x^2}$ is a solution of the functional equation (1.4).

K. Ravi *et al.* [25] investigated the generalized Hyers-Ulam stability of a generalized radical-quadrtic functional equation

$$f\left(\frac{\sqrt{x^2+y^2}}{n}\right)\pm mf\left(\sqrt{x^2+y^2}\right)=(n^2\pm m)\frac{f(x)f(y)}{f(x)+f(y)} \tag{1.5}$$

in Felbin's type fuzzy normed linear spaces.

In this paper, we introduce a new radical type functional equation of the form

$$s\left(x+y+2\sqrt{xy}\right)=s(x)+s(y). \tag{1.6}$$

The radical function $s(x)=\sqrt{x}$ is a solution of the functional equation (1.6). We achieve the general solution of the functional equation (1.6) and investigate its generalized Hyers-Ulam stability, Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by mixed product-sum of powers of norms. We also prove that the functional equation (1.6) is not stable for critical values associated with the results of Th.M. Rassias and J.M. Rassias via counter-examples.

Throughout this paper, let us assume that X be the space of non-negative real numbers.

2. Preliminaries

In this section, we introduce the following definition and theorem in order to proceed further to our main results.

Definition 2.1. *A mapping $s: X \to \mathbb{R}$ is called square root mapping, if the functional equation (1.6) holds for all $x, y \in X$. Note that the mapping s is called square root mapping because the following algebraic identity*

$$\sqrt{x+y+2\sqrt{xy}}=\sqrt{x}+\sqrt{y}$$

holds for all $x, y \in X$ and s satisfies the functional equations

$$s(2^{2n}x)=2^n s(x) \tag{2.1}$$

and

$$s(2^{-2n}x)=2^{-n}s(x) \tag{2.2}$$

for all $x \in X$ and $n \in \mathbb{N}$.

Theorem 2.2. *Let $s: X \to \mathbb{R}$ be a square root mapping satisfying (1.6). Then s satisfies the general functional equations (2.1) and (2.2) for all $x \in X$ and $n \in \mathbb{N}$.*

Proof. Letting $x=0$ and $y=0$ in (1.6), we obtain $s(0)=0$. Plugging (x,y) in (x,x) in (1.6), one finds

$$s(2^2x)=2s(x) \tag{2.3}$$

for all $x \in X$. Now, replacing x by $4x$ in (2.3), one obtains

$$s(2^4x)=2^2s(x) \tag{2.4}$$

for all $x \in X$. Again replacing x by $4x$ in (2.3), we get $s(2^6x)=2^3s(x)$, for all $x \in X$. Proceeding further and using induction on a positive integer n, we achive (2.1). Next, considering (x,y) as $\left(\frac{x}{4},\frac{x}{4}\right)$ in (1.6), one obtains

$$s(2^{-2}x)=2^{-1}s(x) \tag{2.5}$$

for all $x \in X$. Now, letting x as $\frac{x}{4}$ in (2.6), we obtain$s(2^{-6}x)=2^{-3}s(x)$, for all $x \in X$. Proceeding further and using induction on a positive integer n, we obtain (2.2), which completes the proof.

3. General Solution of Functional Equation (1.6)

In this section, we obtain the general solution of functional equation (1.6) in the setting of space of real numbers.

Theorem 3.1. *A mapping $f: X \to \mathbb{R}$ satisfies* (1.6) *if and only if there exists an identity mapping $I: X \to \mathbb{R}$ such that*

$$f(x)=\sqrt{I(x)}, \qquad \textit{for all } x \in X.$$

Proof. Let f satisfy equation (1.6). Then f is a square root mapping and $f(x)=\sqrt{x}$, for all $x \in X$. If I is an identity mapping, then

$$\sqrt{I(x)}=\sqrt{x}=f(x), \text{ for all } x \in X.$$

Conversely, assume that there exists an identity mapping $I: X \to \mathbb{R}$ such that $f(x) = \sqrt{I(x)}$, for all $x \in X$. Hence

$$\begin{aligned} f\left(x+y+2\sqrt{xy}\right) &= \sqrt{I\left(x+y+2\sqrt{xy}\right)} \\ &= \sqrt{x+y+2\sqrt{xy}} \\ &= \sqrt{\left(\sqrt{x}+\sqrt{y}\right)^2} \\ &= \sqrt{x}+\sqrt{y}, \text{ for all } x, y \in X, \end{aligned}$$

which completes the proof of Theorem 3.1.

4. Generalized Hyers-Ulam Stability of Equation (1.6)

In this section, we investigate the generalized Hyers-Ulam stability of the functional equation (1.6) in the setting of space of real numbers. We also extend the stability results in the spirit of Hyers, Th.M. Rassias and J.M. Rassias.

Theorem 4.1. *Let $\phi: X \times X \to \mathbb{R}$ be a mapping satisfying*

$$\Phi(x,y) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi(4^i x, 4^i y) < \infty \qquad (4.1)$$

for all $x, y \in X$. Let $f: X \to \mathbb{R}$ be a mapping such that

$$\left|f\left(x+y+2\sqrt{xy}\right) - f(x) - f(y)\right| \le \phi(x,y) \qquad (4.2)$$

for all $x, y \in X$. Then there exists a unique square root mapping $s: X \to \mathbb{R}$ which satisfies (1.6) and the inequality

$$|f(x) - s(x)| \le \Phi(x,x) \qquad (4.3)$$

for all $x \in X$. The mapping $s(x)$ is defined by

$$s(x) = \lim_{n\to\infty} \frac{1}{2^n} f(4^n x) \qquad (4.4)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Plugging (x,y) into (x,x) in (4.2) and dividing by 2, we obtain

$$\left|\frac{1}{2} f(4x) - f(x)\right| \le \frac{1}{2}\phi(x,x) \qquad (4.5)$$

for all $x \in X$. Now, replacing x by $4x$ in (4.5), dividing by 2 and summing the resulting inequality with (4.5), one finds

$$\left|\frac{1}{2^2} f(4^2 x) - f(x)\right| \le \sum_{i=0}^{1} \frac{1}{2^{i+1}} \phi(4^i x, 4^i x)$$

for all $x \in X$. Using induction arguments, we conclude that

$$\begin{aligned} \left|\frac{1}{2^n} f(4^n x) - f(x)\right| &\le \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \phi(4^i x, 4^i x) \\ &\le \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \phi(4^i x, 4^i x) \qquad (4.6) \end{aligned}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{2^{-n} f(4^n x)\}$, replace x by $4^m x$ in (4.6) and multiply by 2^{-m}, we find that for $n > m > 0$

$$\begin{aligned} &|2^{-m} f(4^m x) - 2^{-n-m} f(4^{n+m} x)| \\ &\quad = 2^{-m} |f(4^m x) - 2^{-n} f(4^{n+m} x)| \\ &\quad \le \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^{m+i}} \phi(4^{m+i} x, 4^{m+i} x) \\ &\quad \to 0 \text{ as } m \to \infty. \end{aligned}$$

This shows that the sequence $\{2^{-n} f(4^n x)\}$ is a Cauchy sequence. Allow $n \to \infty$ in (4.6), we arrive at (4.3). To show that s satisfies (1.6), replacing (x,y) by $(4^n x, 4^n y)$ in (4.2) and multiplying by 2^{-n}, we obtain

$$\begin{aligned} &2^{-n}\left|f\left(4^n\left(x+y+2\sqrt{xy}\right)\right) - f(4^n x) - f(4^n y)\right| \\ &\quad \le 2^{-n}\phi(4^n x, 4^n y) \qquad (4.7) \end{aligned}$$

for all $x, y \in X$. Allowing $n \to \infty$ in (4.7), we see that s satisfies (1.6) for all $x, y \in X$. To prove s is unique square root mapping satisfying (1.6), let us assume $s': X \to \mathbb{R}$ be another square root mapping which satisfies (1.6) and the inequality (4.3). Clearly s' and s satisfy (2.1) and using (4.3), we have

$$\begin{aligned} |s'(x) - s(x)| &= 2^{-n}|s'(4^n x) - s(4^n x)| \\ &\le 2^{-n}(|s'(4^n x) - f(4^n x)| + |f(4^n x) - s(4^n x)|) \\ &\le \sum_{i=0}^{\infty} \frac{1}{2^{n+i}} \phi(4^{n+i} x, 4^{n+i} x) \qquad (4.8) \end{aligned}$$

for all $x \in X$. Allowing $n \to \infty$ in (4.8) and using (4.1), we find that s is unique. This completes the proof of Theorem 4.1.

Theorem 4.2. *Let $\phi: X \times X \to \mathbb{R}$ be a mapping satisfying*

$$\Phi(x,y) = \sum_{i=0}^{\infty} 2^i \phi\left(\frac{x}{4^{i+1}}, \frac{y}{4^{i+1}}\right) < \infty$$

for all $x, y \in X$. Let $f: X \to \mathbb{R}$ be a mapping such that (4.2) *holds for all $x, y \in X$. Then there exists a unique square root mapping $s: X \to \mathbb{R}$ which satisfies* (1.6) *and the inequality*

$$|f(x) - s(x)| \le \Phi(x,x)$$

for all $x \in X$. The mapping $s(x)$ is defined by

$$s(x) = \lim_{n\to\infty} 2^n f(4^{-n} x)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Plugging (x,y) into $\left(\frac{x}{4}, \frac{x}{4}\right)$ in (4.2), one finds

$$\left|f(x) - 2f\left(\frac{x}{4}\right)\right| \le \phi\left(\frac{x}{4}, \frac{x}{4}\right) \qquad (4.9)$$

for all $x \in X$. Now, replacing x by $\frac{x}{4}$ in (4.9), multiplying by 2 and summing the resulting inequality with (4.9), one gets

$$\left|f(x) - 2^2 f\left(\frac{x}{4^2}\right)\right| \le \sum_{i=0}^{1} 2^i \phi\left(\frac{x}{4^{i+1}}, \frac{x}{4^{i+1}}\right)$$

for all $x \in X$. Proceeding further and using induction on a positive integer n, we conclude that

$$\begin{aligned} |f(x) - 2^n f(4^{-n} x)| &\le \sum_{i=0}^{n-1} 2^i \phi\left(\frac{x}{4^{i+1}}, \frac{x}{4^{i+1}}\right) \\ &\le \sum_{i=0}^{\infty} 2^i \phi\left(\frac{x}{4^{i+1}}, \frac{x}{4^{i+1}}\right) \end{aligned}$$

for all $x \in X$. The rest of the proof is obtained by similar arguments as in Theorem 4.1.

Corollary 4.3. *Let $f: X \to \mathbb{R}$ be a mapping for which there exists a constant $c \geq 0$, independent of x, y such that the functional inequality*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \leq c$$

holds for $x, y \in X$. Then $s: X \to \mathbb{R}$ defined by

$$s(x) = \lim_{n\to\infty} \frac{1}{2^n} f(4^{-n}x)$$

is a square root mapping satisfying the functional equation (1.6) *and*

$$|f(x)-s(x)| \leq c$$

for all $x \in X$ and $n \in \mathbb{N}$.

Proof. Taking $\phi(x, y) = c$, for all $x, y \in X$ in Theorem 4.1, we get $\phi(x, x) = c$. From (4.3), one finds

$$|f(x)-s(x)| \leq \frac{c}{2}\sum_{i=0}^{\infty}\frac{1}{2^i} \leq c$$

for all $x \in X$.

Corollary 4.4. *Let $c_1 \geq 0$ be fixed and $p \neq \frac{1}{2}$. If a mapping $f: X \to \mathbb{R}$ satisfies the inequality*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \leq c_1(|x|^p+|y|^p)$$

for all $x, y \in X$, then there exists a unique square root mapping $s: X \to \mathbb{R}$ satisfying the functional equation (1.6) *and*

$$|f(x)-s(x)| \leq \frac{2c_1}{|2-4^p|}|x|^p, \text{ for all } x \in X.$$

Proof. Considering $\phi(x, y) = c_1(|x|^p+|y|^p)$, for all $x, y \in X$ in Theorems 4.1 and 4.2, we arrive at the required results.

Corollary 4.5. *Let $f: X \to \mathbb{R}$ be a mapping. If there exist $a, b: \rho = a+b \neq \frac{1}{2}$ and $c_2 \geq 0$ such that*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \leq c_2|x|^a|y|^b$$

for all $x, y \in X$, then there exists a unique square root mapping $s: X \to \mathbb{R}$

$$|f(x)-s(x)| \leq \frac{c_2}{|2-4^\rho|}|x|^\rho, \text{ for all } x \in X.$$

Proof. The proof is obtained by letting $\phi(x, y) = c_2|x|^a|y|^b$, for all $x, y \in X$ in Theorems 4.1 and 4.2.

Corollary 4.6. *Let $f: X \to \mathbb{R}$ be a mapping. If there exist $c_3 \geq 0$ and $\alpha > 0$ with $\alpha \neq \frac{1}{4}$ such that*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \leq c_3\left(|x|^\alpha|y|^\alpha+(|x|^{2\alpha}+|y|^{2\alpha})\right)$$

for all $x, y \in X$, then there exists a unique square root mapping $s: X \to \mathbb{R}$ satisfying the functional equation (1.6) *and the inequality*

$$|f(x)-s(x)| \leq \frac{3c_3}{|2-4^{2\alpha}|}|x|^{2\alpha}, \text{for all } x \in X.$$

Proof. By choosing $\phi(x, y) = c_3\left(|x|^\alpha|y|^\alpha+(|x|^{2\alpha}+|y|^{2\alpha})\right)$, for all $x, y \in X$ in Theorems 4.1 and 4.2, the proof is obtained.

5. Counter-Examples for Singular Cases

In this section, we present counter-examples to prove that the functional equation (1.6) isnot stable for singular cases in Corollaries 4.4, 4.5 and 4.6.

Using the idea of Z. Gajda [9], we construct the following counter-example to prove that the Corollary 4.4 is false for $p = \frac{1}{2}$.

Define a mapping $f: X \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty}\frac{\varphi(4^n x)}{2^n} \tag{5.1}$$

for all $x \in X$, where the function $\varphi: X \to \mathbb{R}$ is given by

$$\varphi(x) = \begin{cases} \mu\sqrt{x}, \text{for all } x, y \in (0,1) \\ \mu, \text{otherwise.} \end{cases} \tag{5.2}$$

Then the function f serves as a counter-example for $p = \frac{1}{2}$ as presented in the following theorem.

Theorem 5.1. *The mapping f defined above satisfies*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \leq 12\mu\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right) \tag{5.3}$$

for all $x, y \in X$. Therefore there do not exist a square root mapping $s: X \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x)-s(x)| \leq \beta|x|^{\frac{1}{2}} \tag{5.4}$$

for all $x \in X$.

Proof. $|f(x)| \leq \sum_{n=0}^{\infty}\frac{|\varphi(4^n x)|}{|2^n|} \leq \sum_{n=0}^{\infty}\frac{\mu}{2^n} = \mu\left(1-\frac{1}{2}\right)^{-1} = 2\mu$. Hence f is bounded by 2μ. If $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} \geq \frac{1}{2}$, then the left-hand side of (5.3) is less than 6μ. Now, suppose that$0 < |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < \frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \leq |x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < \frac{1}{2^k}. \tag{5.5}$$

Hence $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}} < \frac{1}{2^k}$ implies

$$4^{\frac{k}{2}}|x|^{\frac{1}{2}}+4^{\frac{k}{2}}|y|^{\frac{1}{2}} < 1$$

or

$$4^{\frac{k}{2}}x^{\frac{1}{2}}+4^{\frac{k}{2}}y^{\frac{1}{2}} < 1$$

or

$$4^{\frac{k-1}{2}}x^{\frac{1}{2}} < \frac{1}{2}, 4^{\frac{k-1}{2}}y^{\frac{1}{2}} < \frac{1}{2}$$

and consequently

$$4^{k-1}(x), 4^{k-1}(y), 4^{k-1}(x+y+2\sqrt{xy}) \in (0,1).$$

Therefore, for each value of $n = 0,1,2,\ldots,k-1$, we obtain

$$4^n(x), 4^n(y), 4^n(x+y+2\sqrt{xy}) \in (0,1)$$

and $\varphi\left(4^n(x+y+2\sqrt{xy})\right) - \varphi(4^n x) - \varphi(4^n y) = 0$

for $n = 0,1,2,\ldots,k-1$. Using (5.5) and the definition of f, we obtain

$$\begin{aligned}
&|f(x+y+2\sqrt{xy}) - f(x) - f(y)| \\
&\le \sum_{n=0}^{\infty} \frac{1}{2^n}\left|\varphi\left(4^n(x+y+2\sqrt{xy})\right) - \varphi(4^n x) - \varphi(4^n y)\right| \\
&\le \sum_{n=k}^{\infty} \frac{1}{2^n}\left|\varphi\left(4^n(x+y+2\sqrt{xy})\right) - \varphi(4^n x) - \varphi(4^n y)\right| \\
&\le \sum_{n=k}^{\infty} \frac{3\mu}{2^n} \\
&\le \frac{12\mu}{2^{k+1}} \le 12\mu\left(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}}\right).
\end{aligned}$$

Therefore, the inequality (5.3) holds true. We claim that the square root functional equation (1.6) is not stable for $p = \frac{1}{2}$ in Corollary 4.4.

Assume that there exists a square root mapping $s: X \to \mathbb{R}$ satisfying (5.4). Therefore, we have

$$|f(x)| \le (\beta+1)|x|^{\frac{1}{2}}. \qquad (5.6)$$

However, we can choose a positive integer m with $m\mu > \beta + 1$. If $x \in (0, 4^{1-m})$, then $4^n x \in (0,1)$ for all $n = 0,1,2,\ldots,m-1$ and therefore

$$|f(x)| = \sum_{n=0}^{\infty} \frac{\varphi(4^n x)}{2^n} \ge \sum_{n=0}^{m-1} \frac{2^n \mu\sqrt{x}}{2^n} = m\mu\sqrt{x} > (\beta+1)\sqrt{x}$$

which contradicts (5.6). Therefore, the square root functional equation (1.6) is not stable for $p = \frac{1}{2}$ in Corollary 4.4.

The following example illustrates that the functional equation (1.6) is not stable when $\rho = a + b = \frac{1}{2}$ in Corollary 4.5.

Inspired by the counter-example provided by P. Gavruta in [11], we present the following example which establishes that the Corollary 4.5 is false for $\rho = a + b = \frac{1}{2}$.

Define a mapping $f: X \to \mathbb{R}$ by

$$f(x) = \begin{cases} \sqrt{x}\, \ln|x| \; if\; x \ne 0 \\ 0 \; if\; x = 0. \end{cases} \qquad (5.7)$$

Then the function f defined in (5.7) turns out to be a counter-example for $\rho = \frac{1}{2}$ as proved in the following theorem.

Theorem 5.2. *Let a be such that $0 < a < \frac{1}{2}$. Then there is a function $f: X \to \mathbb{R}$ and a constant $c_2 \ge 0$ satisfying*

$$|f(x+y+2\sqrt{xy}) - f(x) - f(y)| \le c_2|x|^a|y|^{\left(\frac{1}{2}-a\right)} \qquad (5.8)$$

for all $x, y \in X$ and

$$sup_{x\ne 0}\frac{|f(x)-s(x)|}{\sqrt{x}} = +\infty \qquad (5.9)$$

for every square root mapping $s: X \to \mathbb{R}$.

Proof. From the relation (5.9), it follows that

$$\begin{aligned}
sup_{x\ne 0}\frac{|f(x)-s(x)|}{\sqrt{x}} &\ge sup_{n\in\mathbb{N}, n\ne 0}\frac{|f(n)-s(n)|}{\sqrt{n}} \\
&= sup_{n\in\mathbb{N}, n\ne 0}\frac{\sqrt{n}\ln|n| - \sqrt{n}s(1)}{\sqrt{n}} \\
&= sup_{n\in\mathbb{N}, n\ne 0}|\ln|n| - s(1)| = \infty.
\end{aligned}$$

We have prove (5.8) is true.

Case 1. If $x = y = 0$ in (5.8), then

$$|f(0) - f(0) - f(0)| = |f(0)| = 0.$$

Case 2. If $x, y > 0$, then the left-hand side of (5.8) becomes

$$\begin{aligned}
&|f(x+y+2\sqrt{xy}) - f(x) - f(y)| \\
&= \left(\sqrt{x+y+2\sqrt{xy}}\right)\ln(x+y+2\sqrt{xy}) \\
&\qquad - \sqrt{x}\ln(x) - \sqrt{y}\ln(y) \\
&= (\sqrt{x}+\sqrt{y})\ln(x+y+2\sqrt{xy}) - \sqrt{x}\ln(x) - \sqrt{y}\ln(y) \\
&= \sqrt{x}\ln\left(\frac{x+y+2\sqrt{xy}}{x}\right) + \sqrt{y}\ln\left(\frac{x+y+2\sqrt{xy}}{y}\right) \\
&= \sqrt{x}\ln\left(1+\frac{y}{x}+2\sqrt{\frac{y}{x}}\right) + \sqrt{y}\ln\left(1+\frac{x}{y}+2\sqrt{\frac{x}{y}}\right).
\end{aligned}$$

We have to prove that there exists a constant $c_2 \ge 0$ such that

$$\sqrt{x}\ln\left(1+\frac{y}{x}+2\sqrt{\frac{y}{x}}\right) + \sqrt{y}\ln\left(1+\frac{x}{y}+2\sqrt{\frac{x}{y}}\right) \le c_2|x|^a|y|^{\left(\frac{1}{2}-a\right)} \qquad (5.10)$$

for all $x, y > 0$. With $\frac{y}{x} = t$, the inequality (5.10) is equivalent to the inequality

$$\frac{1}{t^{\left(\frac{1}{2}-a\right)}}\ln(1+t+2\sqrt{t}) + t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right) \le c_2, t > 0. \qquad (5.11)$$

By using L' Hospital rule,

$$\lim_{t\to 0} t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right) = \frac{1}{a}\lim_{t\to 0}\frac{t^a(2+\sqrt{t})}{t+2\sqrt{t}+1} = 0$$

and $\lim_{t\to\infty} t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right) = \frac{1}{a}\lim_{t\to\infty}\frac{t^a(2+\sqrt{t})}{t+2\sqrt{t}+1} = 0$

and since the function

$$t \longrightarrow t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right)$$

is continuous on $(0,\infty)$, it follows that there is a constant $c_2 = c_2(a) > 0$ such that

$$t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right) \le c_2, t > 0. \qquad (5.12)$$

If we replace t by $\frac{1}{t}$and a by $\left(\frac{1}{2}-a\right)$ in (5.12), it follows that

$$\frac{1}{t^{\left(\frac{1}{2}-a\right)}}\ln\left(1+t+2\sqrt{t}\right) \le c_2\left(\frac{1}{2}-a\right), t>0. \quad (5.13)$$

From (5.12) and (5.13), we conclude that
$$\frac{1}{t^{\left(\frac{1}{2}-a\right)}}\ln\left(1+t+2\sqrt{t}\right)+t^a\ln\left(1+\frac{1}{t}+\frac{2}{\sqrt{t}}\right) \le \frac{c_2}{2} \le c_2,\ t>0.$$

That is, (5.11) holds and hence (5.8) holds.
In order to show that the functional equation (1.6) is not stable if $\alpha=\frac{1}{4}$ in Corollary 4.6, consider the functions f and φ defined in (5.1) and (5.2) respectively. Then the function f defined in (5.1) becomes a counter-example for $\alpha=\frac{1}{4}$ as presented in the following theorem.

Theorem 5.3. *The mapping f defined in (5.1) satisfies the inequality*

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right| \le 12\mu\left(|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)\right) \quad (5.14)$$

for all $x,y\in X$. Therefore there do not exist a square root mapping $s: X\to\mathbb{R}$ and a constant $\beta>0$ such that

$$|f(x)-s(x)| \le \beta|x|^{\frac{1}{2}} \quad (5.15)$$

for all $x\in X$.

Proof. $|f(x)| \le \sum_{n=0}^{\infty}\frac{|\varphi(4^n x)|}{|2^n|} \le \sum_{n=0}^{\infty}\frac{\mu}{2^n}=\mu\left(1-\frac{1}{2}\right)^{-1}=2\mu$. Hence f is bounded by 2μ. If $\left(|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)\right)\ge\frac{1}{2}$, then the left-hand side of (5.14) is less than 6μ. Now, suppose that $0<\left(|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)\right)<\frac{1}{2}$. Then there exists a positive integer k such that

$$\frac{1}{2^{k+1}} \le |x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)<\frac{1}{2^k}. \quad (5.16)$$

Hence $|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)<\frac{1}{2^k}$ implies
$$2^k|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+2^k\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)<1$$
or
$$4^{\frac{k}{2}}|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(4^{\frac{k}{2}}|x|^{\frac{1}{2}}+4^{\frac{k}{2}}|y|^{\frac{1}{2}}\right)<1$$
or
$$4^{\frac{k}{4}}|x|^{\frac{1}{4}}4^{\frac{k}{4}}|y|^{\frac{1}{4}}+\left(4^{\frac{k}{2}}|x|^{\frac{1}{2}}+4^{\frac{k}{2}}|y|^{\frac{1}{2}}\right)<1$$
or
$$4^{\frac{k}{4}}x^{\frac{1}{4}}<1, 4^{\frac{k}{4}}y^{\frac{1}{4}}<1, 4^{\frac{k}{2}}x^{\frac{1}{2}}, 4^{\frac{k}{2}}y^{\frac{1}{2}}<1$$
or
$$4^{\frac{k-1}{4}}x^{\frac{1}{4}}<\frac{1}{\sqrt{2}}, 4^{\frac{k-1}{4}}y^{\frac{1}{4}}<\frac{1}{\sqrt{2}}, 4^{\frac{k-1}{2}}x^{\frac{1}{2}}<\frac{1}{2}, 4^{\frac{k-1}{2}}y^{\frac{1}{2}}<\frac{1}{2}$$

and consequently

$$4^{k-1}(x), 4^{k-1}(y), 4^{k-1}\left(x+y+2\sqrt{xy}\right)\in(0,1).$$

Therefore, for each value of $n=0,1,2,\dots,k-1$, we obtain

$$4^n(x), 4^n(y), 4^n\left(x+y+2\sqrt{xy}\right)\in(0,1)$$
and $\varphi\left(4^n\left(x+y+2\sqrt{xy}\right)\right)-\varphi(4^n x)-\varphi(4^n y)=0$
for $n=0,1,2,\dots,k-1$. Using (5.16) and the definition of f, we obtain

$$\left|f\left(x+y+2\sqrt{xy}\right)-f(x)-f(y)\right|$$
$$\le \sum_{n=0}^{\infty}\frac{1}{2^n}\left|\varphi\left(4^n\left(x+y+2\sqrt{xy}\right)\right)-\varphi(4^n x)-\varphi(4^n y)\right|$$
$$\le \sum_{n=k}^{\infty}\frac{1}{2^n}\left|\varphi\left(4^n\left(x+y+2\sqrt{xy}\right)\right)-\varphi(4^n x)-\varphi(4^n y)\right|$$
$$\le \sum_{n=k}^{\infty}\frac{3\mu}{2^n}$$
$$\le \frac{12\mu}{2^{k+1}} \le 12\mu\left(|x|^{\frac{1}{4}}|y|^{\frac{1}{4}}+\left(|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}\right)\right).$$

Therefore, the inequality (5.14) holds true. The rest of the proof is similar to that of Theorem 5.1.

6. Conclusion
Thus we have obtained the general solution and proved the Ulam stability of the functional equation (1.6) pertinent to D.H. Hyers, Th.M. Rassias, J.M. Rassias and P. Gavruta. Further, we have shown that the functional equation (1.6) is not stable for singular cases by illustrating counter-examples.

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