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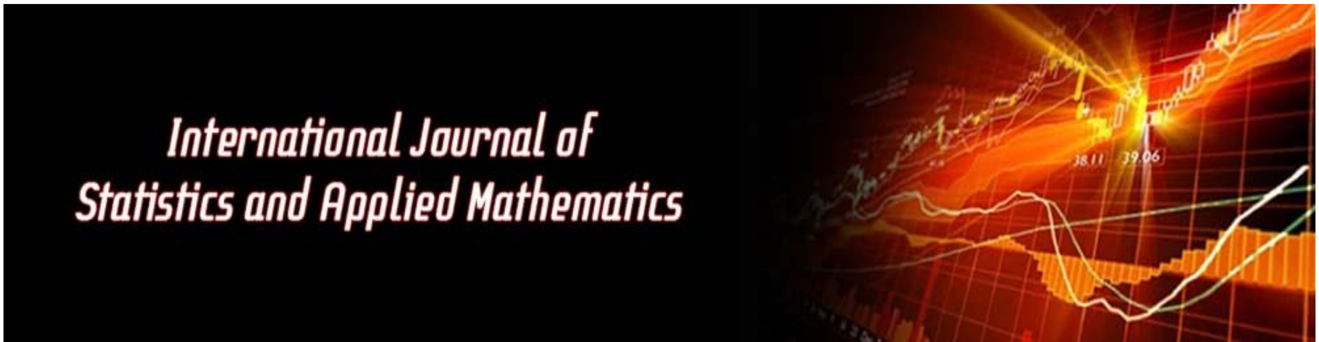

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Hyers – Ulam stability of linear difference equations of first order

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Abstract
We prove the Hyers – Ulam stability of linear difference equations of first order of the form $\phi(t)\Delta y(t) = y(t)$.

Keywords: Hyes – Ulam stability, difference equation, first order.

1. Introduction

The theory of difference equations and their applications have been receiving intensive attention. See, for example [1-3] and the references cited therein.
In this paper we consider the first order difference equation of the form

$$\phi(t)\Delta y(t) = y(t), \ t\in I \tag{1}$$

where, $I = N(a) = \{a, a+1, a+2, \dots\}$, (a is a fixed nonnegative integer), Δ is the forward difference operator defined by $\Delta y(t) = y(t+1) - y(t)$. Assume further that $\phi: I \to R$ is a given function. By a solution of equation (1) we mean a sequence $\{y(t)\}$ which is defined for $t \in I$, and satisfies equation (1). S.M Jung the author [4] who investigated the Hyers –Ulam stability of linear differential equation of first order $\phi(t)y'(t) = y(t)$, which was the improvement of the papers [5, 6]. Indeed they dealt with the Hyers Ulam stability of differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$.
The aim of this paper is to investigate the Hyers –ulam stability of the linear difference equation of first order (1). More precisely, we prove that if $\phi(t) > 0$ or $\phi(t) < -1$ hold for all $t \in I$ and further if the function y(t) satisfies $|\phi(t)\Delta y(t) - y(t)| < \epsilon \ \forall \ t\in I$ then there exists a real number c such that $\left|y(t) - c\sum_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\right| \le \epsilon$ for any $t \in I$.

2. Preliminaries

Following an idea of Scon – Mo Jung [4] we prove the following lemma

Lemma:2.1 Assume that a function $z: I \to R$ is given, The inequality $z(t) \le \emptyset(t)\Delta z(t)$ is true for $t \in I$, if and only if there exists a function $\alpha: I \to R$ such that $\Delta\alpha(t)\emptyset(t) > 0$ and $z(t) = \alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} \ \forall \ t \in I$.

Proof: Assume the inequality $z(t) \le \emptyset(t)\Delta z(t)$ holds true for all $t \in I$. Let us define function $\alpha: I \to R$ such that

$$\alpha(t) = z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}$$

then,

$$\Delta\alpha(t) = \Delta\left(\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}\right)z(t) + \prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}\Delta z(t)$$

$$= \prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}[\Delta z(t)] + z(t+1)\frac{\emptyset(t)}{1+\emptyset(t)}$$

$$\emptyset(t)\Delta\alpha(t)=\emptyset(t)\Delta z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}+z(t+1)\frac{\emptyset(t)^2}{1+\emptyset(t)}$$

Since by hypothesis $\emptyset(t)\Delta z(t) \geq z(t)$

$\emptyset(t)\Delta\alpha(t) \geq z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}+z(t+1)\frac{\emptyset(t)^2}{1+\emptyset(t)} > 0$ and

$\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} = z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} = z(t).$

Conversely, assume that there exists a function $\alpha: I \rightarrow R$ such that $\emptyset(t)\Delta\alpha(t) \geq 0$ for each $t \in I$. Let us define a function $z: I \rightarrow R$ by,

$z(t)=\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$, then,

$\Delta z(t)=\Delta\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}+\alpha(t+1)\Delta\left(\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\right)$

Therefore $\emptyset(t)\Delta z(t)=\emptyset(t)\Delta\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}+\alpha(t+1)\left[\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\right]$

Since by hypothesis $\emptyset(t)\Delta\alpha(t) \geq 0$, we have

$\emptyset(t)\Delta z(t) \geq \alpha(t+1)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} \geq z(t), \forall t \in I$

which proves lemma.

Lemma:2.2 Assume that a function $z: I \rightarrow R$ is given, The inequality $z(t) \geq \emptyset(t)\Delta z(t)$ holds true for any $t \in I$, if and only if there exists a function $\beta: I \rightarrow R$ such that $\Delta\beta(t)\emptyset(t) < 0$ and $z(t)=\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}, \forall t \in I.$

Proof: Assume the inequality $z(t) \geq \emptyset(t)\Delta z(t)$ holds for all $t \in I$. Let us define $\beta: I \rightarrow R$ to be $\beta(t)=\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}z(t)$ then,

$\Delta\beta(t)=\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}\Delta z(t)+z(t+1)\Delta\left(\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}\right)$

$\emptyset(t)\Delta\beta(t)=\emptyset(t)\Delta z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}-z(t+1)\left[\prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}\right]$

Since, $\emptyset(t)\Delta z(t) \leq z(t)$

$\emptyset(t)\Delta\beta(t) \leq z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}-z(t+1)\left[\prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}\right] < 0$, and

$\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} = z(t)\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} = z(t).$

Conversely, assume that $\Delta\beta(t)\emptyset(t) \leq 0$ for each $t \in I$ and let us define a function

$z: I \rightarrow R$ by $z(t)=\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$ then,

$\emptyset(t)\Delta z(t) = \emptyset(t)\Delta\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}+\beta(t+1)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$

since $\emptyset(t)\Delta\beta(t) \leq 0$,

$\emptyset(t)\Delta z(t) \leq \beta(t+1)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$

Therefore $\emptyset(t)\Delta z(t) \leq z(t)$. Hence the proof.

Theorem: 2.3 Given an $\epsilon > 0$, a function $y: I \rightarrow R$ is a solution of the following inequality

$\left|\emptyset(t)\Delta y(t)-y(t)\right| \leq \epsilon$ for all $t \in I$ (2)

if and only if there exists a function $\alpha: I \rightarrow R$ such that

$y(t)=\epsilon+\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$ (3)

and $0 \leq \Delta\alpha(t)\emptyset(t) \leq 2\epsilon\prod_{s=a}^{t-1}\frac{\emptyset(s)}{1+\emptyset(s)}, \forall t \in I.$ (4)

Proof: First we assume that $y: I \rightarrow R$ is a solution of the inequality (2) then y satisfies

$y(t)-\epsilon \leq \emptyset(t)\Delta y(t) \leq y(t)+\epsilon$ (5)

for each $t \in I$. Define $z(t)=y(t)-\epsilon$ then $\Delta z(t)=\Delta y(t)$, the inequality on the L.H.S of (5), becomes $z(t) \leq \emptyset(t)\Delta z(t)$ holds for every $t \in I$. According to lemma 1.1 there exists a function $\alpha: I \rightarrow R$ such that

$y(t)=\epsilon+\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$ for all $t \in I$. (6)

where α additionally satisfies the condition,

$\Delta\alpha(t)\emptyset(t) > 0$, for any $t \in I$ (7)

Analogously define $z(t)=y(t)+\epsilon$, the inequality on the R.H.S of (5) implies that $z(t) \geq \emptyset(t)\Delta y(t)=\emptyset(t)\Delta z(t)$ holds for any $t \in I$. According to lemma 1.2 there exists a function $\beta: I \rightarrow R$ such that $y(t)+\epsilon=z(t)=\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$ (8)

and $\Delta\beta(t)\emptyset(t) < 0$ for all $t \in I$. (9)

From (6),

$\Delta y(t) = \alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\frac{1}{\emptyset(t)}+\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}\Delta\alpha(t)$ (10)

From (8), $\Delta y(t) = \left(\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\right)\frac{\beta(t)}{\emptyset(t)}+\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}\Delta\beta(t)$ (11)

Also, from (8) and (6)

$\beta(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)} = 2\epsilon+\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$

substituting in (11)

$\Delta y(t) = \Delta\beta(t)\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}+\frac{2\epsilon}{\emptyset(t)}+\frac{\alpha(t)}{\emptyset(t)}\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$

substituting for $\Delta y(t)$ from (10) we get,

$\Delta\alpha(t)\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)} = \Delta\beta(t)\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}+\frac{2\epsilon}{\emptyset(t)}$

$\Delta\alpha(t)=\Delta\beta(t)+\frac{2\epsilon}{\emptyset(t)}\prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}$

$\emptyset(t)\Delta\beta(t)=\emptyset(t)\Delta\alpha(t)-2\epsilon\prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}$

from (7) and (9) we get,

$0 \leq \Delta\alpha(t)\emptyset(t) \leq 2\epsilon\prod_{s=a}^{t}\frac{\emptyset(s)}{1+\emptyset(s)}$, for any $t \in I$, which proves (4).

Conversely, assume that $y: I \rightarrow R$ is given by (3).

$y(t)=\epsilon+\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\ \forall t \in I$

and a function $\alpha: I \rightarrow R$ satisfies

$0 \leq \Delta\alpha(t)\emptyset(t) \leq 2\epsilon\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}$

then from(10),

$$\emptyset(t)\Delta y(t)=\alpha(t)\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}+\emptyset(t)\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}\Delta\alpha(t)$$

Using (3) and (4)

$\emptyset(t)\Delta y(t)-y(t)=\emptyset(t)\Delta\alpha(t)\prod_{s=a}^{t}\frac{1+\emptyset(s)}{\emptyset(s)}-\epsilon.$

By (4) and the last equation, we conclude that

$\left|\emptyset(t)\Delta z(t)-y(t)\right| \leq \epsilon$. Which proves the theorem.

3. Hyers – Ulam Stability of difference equation (1)

In the following theorem, we prove the Hyers-Ulam stability of the difference equation (1).

Theorem: 3.1 If either $\emptyset(t) > 0$ holds for all $t \in I$, or $1+\emptyset(t) < 0$ holds for all $t \in I$, and if a function $y: I \rightarrow R$ satisfies the inequality (2) then there exists a real number c such that $\left|y(t)-c\prod_{s=a}^{t-1}\frac{1+\emptyset(s)}{\emptyset(s)}\right| \leq \epsilon$ for any $t \in I$. (12)

Proof: First we assume that $\emptyset(t) > 0$ holds for all $t \in I$ and a function $y: I \rightarrow R$ satisfies the inequality (2) for all $t \in I$. Let $\alpha: I \rightarrow R$ is a function such that (3) and (4) hold, using Theorem 2.3

$y(t) = \epsilon + \alpha(t) \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)}$ and

$$0 \leq \Delta\alpha(t)(\emptyset(t)) \leq 2\epsilon \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1+\emptyset(s)}$$

Define $c = \lim_{t\rightarrow a} \alpha(t)$. Also we use $\prod_{s=a}^{a-1} \frac{1+\emptyset(s)}{\emptyset(s)} = 1$.
Now we can divide the above inequality by $-(\emptyset(t))$ we get,

$$0 \geq -\Delta\alpha(t) \geq \frac{-2\epsilon}{(\emptyset(t))} \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1+\emptyset(s)}$$

Since $1+\emptyset(t) > \emptyset(t)$

Taking the summation on both sides, we get

$$0 \geq -\sum \Delta\alpha(t) \geq 2\epsilon \sum \frac{-1}{(1+\emptyset(t))} \prod_{s=a}^{t-1} \frac{\emptyset(s)}{1+\emptyset(s)}.$$

$$0 \geq -\alpha(t)+c \geq 2\epsilon \left(\prod_{s=a}^{t-1} \frac{\emptyset(s)}{1+\emptyset(s)}\right)$$

$$-\epsilon \geq -\alpha(t) \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} + c \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} - \epsilon \geq \epsilon$$

$$-\epsilon \leq \alpha(t) \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} + \epsilon - c \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} \leq \epsilon$$

$$-\epsilon \leq y(t) - c \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} \leq \epsilon,$$

$\left| y(t) - c \prod_{s=a}^{t-1} \frac{1+\emptyset(s)}{\emptyset(s)} \right| \leq \epsilon$ for any $t \in I$, which proves (12).

If we assume that $1 + \emptyset(t) < 0$ holds true for all $t \in I$, then the proof is similar to the above procedure, hence we omit it.

4. Remark: Here, we notice that $y(a) \prod_{s=a}^{t-1} \emptyset(s)$ is the general solution of the difference equation $\emptyset(t)\Delta y(t) = y(t)$, where $\emptyset(s) = \frac{1+\emptyset(s)}{\emptyset(s)}$.

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