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Two step estimation in burr type X distribution

MS Ravikumar and RRL Kantam

Abstract
The two parameter Burr type X distribution is considered. The well known classical method – maximum likelihood (ML) estimation of both the parameters is attempted resulting in iterative solutions of the parameters. Hence, we propose reasonably efficient and computationally simple/analytical, estimation procedure called two-step estimation and present a comparative study of the efficiency of the proposed procedure in relation to the ML method.

Keywords: Burr type X distribution, Log likelihood, Asymptotic Variance, MLE, order statistics

1. Introduction

In reliability studies exponential distribution is the central model for the study of any phenomenon through a probabilistic approach. Extension of this model into two different directions yields two popular models called the Weibull distribution and the gamma distribution. Between these two, the Weibull distribution is more frequently applied model in any practical situation concerning reliability studies. Both Weibull and gamma distributions involve a shape parameter, in the sense that the shape of the frequency curve of these models changes according as the change in the values of the shape parameters. In particular, for the Weibull distribution its shape parameter classifies it into an Increasing Failure Rate (IFR) model or a Decreasing Failure Rate (DFR) model or Constant Failure Rate (CFR) model (the well-known exponential model) as its special cases. The natural phenomenon in reliability is "The aging concept" – indicated by increasing instantaneous failure probability with age of the product. A specific case of Weibull distribution exhibiting aging effect with an integer valued shape parameter is known as "The Rayleigh distribution". Its cumulative distribution function (CDF), probability density function (PDF) and hazard function are given by

$$F(x) = 1 - e^{-x^2} \tag{1.1}$$
$$f(x) = 2xe^{-x^2} \tag{1.2}$$
$$h(x) = 2x \tag{1.3}$$

If F(x) is the cumulative distribution function of a positive valued random variable, then $[F(x)]^k$, k>0 also satisfies all the requirements for the cumulative distribution function of another positive valued random variable. It can be interpreted as the failure probability of a parallel system of k- components whose life times are independently and identically distributed random variables, each with a common CDF – F(x) if k is an integer. Exploring this concept to non-integer values of k also, many researchers in the recent past made extensive studies on models of the type $[F(x)]^k$ generated by a basic well known model F(x). Such new models are named as exponentiated models by some authors and generalised models by some authors. For instance if the basic F(x) is exponential, $[F(x)]^k$ is named as generalised exponential (Gupta & Kundu- 1999) [3], if F(x) is Weibull, $[F(x)]^k$ is named as exponentiated Weibull (Mudholkar and Srivastava – 1993). Banking on this notion, generalised Rayleigh distribution was studied by Raqab and Kundu (2006) [5], as a process of revisit to Burr type X distribution. Its cumulative distribution function (CDF), probability density function (PDF) and hazard function are given by

$$F(x;k) = (1 - e^{-x^2})^k;\ x>0,\ k>0 \tag{1.4}$$
$$f(x;k) = 2kxe^{-x^2}(1 - e^{-x^2})^{k-1};\ x>0,\ k>0 \tag{1.5}$$

$$h(x;k) = \frac{2kxe^{-x^2}(1-e^{-x^2})^{(k-1)}}{1-(1-e^{-x^2})^k} \qquad (1.6)$$

Burr (1942) [1] has suggested a number of forms of cumulative distribution functions that might be useful in modeling various practical situations. In all, he suggested twelve models as listed below.

$$\left.\begin{array}{ll}
1. & F(x) = x\,;\ 0 < x < 1, \\
2. & F(x) = (e^{-x}+1)^{-k}, \\
3. & F(x) = (x^{-c}+1)^{-k};\ 0 < x < \infty, \\
4. & F(x) = \left[\left(\frac{c-x}{x}\right)^{1/c}+1\right]^{-k}\ ;\ 0 < x < c, \\
5. & F(x) = (ce^{-\tan x}+1)^{-k};\ -\frac{\pi}{2} < x < \frac{\pi}{2}, \\
6. & F(x) = \left(ce^{-k\sinh x}+1\right)^{-k}, \\
7. & F(x) = 2^{-k}(1+\tanh x)^k, \\
8. & F(x) = \left(\frac{2}{\pi}\, arc\tan e^x\right)^k, \\
9. & F(x) = 1 - \frac{2}{c[(1+e^x)^k-1]+2}, \\
10. & F(x) = \left(1-e^{-x^2}\right)^k; 0 < x < \infty, \\
11. & F(x) = \left(x - \frac{1}{2\pi}\sin 2\pi x\right)^k\ ;\ 0 < x < 1, \\
12. & F(x) = 1-(1+x^c)^{-k}; 0 < x < \infty.
\end{array}\right\} \qquad (1.7)$$

Thus generalised Rayleigh distribution and Burr type X distributions are one and the same.

In the above models *k* and *c* are the positive parameters involved in the respective models. The first model is the well-known uniform distribution also included by Burr (1942) [1]. Among these twelve forms, the type X and type XII models are most frequently applied by many researchers. Our focus is Burr Type X model, whose expressions are given in equations (1.4), (1.5), and (1.6). If a scale parameter say λ is introduced, the cumulative distribution function, probability density function and hazard function are given by

$$F(x;k,\lambda) = \left(1-e^{-(\lambda x)^2}\right)^k;\ x > 0, k > 0, \lambda > 0, \qquad (1.8)$$

$$f(x;k,\lambda) = 2k\lambda^2 xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{(k-1)}; x > 0, k > 0, \lambda > 0, \qquad (1.9)$$

$$h(x;k,\lambda) = \frac{2k\lambda^2 xe^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{(k-1)}}{1-\left(1-e^{-(\lambda x)^2}\right)^k}. \qquad (1.10)$$

Expressions in (1.4), (1.5) and (1.6) are called standard Burr type X model, those in equations (1.8), (1.9) and (1.10) are called scaled Burr type X or Two parameter Burr type X model.

In this paper estimation of parameters of the Burr type X model is studied. Though this is attempted by other researchers in the past (Kundu and Raqab- 2005 and the references therein) [6] our approach of estimation is different from theirs. We consider the two parameter Burr X distribution one of which is its scale parameter and the other is its shape parameter. We first take up the well-known classical maximum likelihood method of estimation of the parameters from complete sample when both the parameters are unknown. In view of the iterative nature of the solutions of the log likelihood equations, we make a modest beginning of estimating one of the parameters when the other is known. For the classical maximum likelihood estimation approach for both the parameters we derive the elements of the information matrix, asymptotic dispersion matrix, and their computed values in Section - 2 as these are rarely available in published form. We estimate both the parameters *λ* and *k* of Burr type X distribution by using Two Step Estimation method. The asymptotic relative efficiency of two step estimation results are presented in relation to empirical dispersion matrix, asymptotic dispersion matrix in Section - 3.

2. Maximum Likelihood Estimation

The probability density function of the two parameter Burr type X distribution is given by

$$f(x;k,\lambda) = 2k\lambda^2 xe^{-(\lambda x)^2}\left(1-e^{-(\lambda x)^2}\right)^{(k-1)}; x > 0, k > 0, \lambda > 0 \qquad (2.1)$$

Let $x_1<x_2<x_3<\ldots<x_n$ be a complete ordered sample of size *n* drawn from the above distribution (Though for complete samples maximum likelihood estimation does not require ordering of the sample, in order to facilitate presenting the methods to be introduced in Section – 3 which depend on ordered samples we consider ordered complete samples in the beginning itself, for uniformity in the notation.). The log likelihood equations to get the maximum likelihood estimates of *λ* and *k* are given by (after simplification).

$$\frac{\partial logL}{\partial\lambda} = 0 \Rightarrow \frac{2n}{\lambda} - 2\lambda\sum_{i=1}^{n} x_i^2 + 2\lambda(k-1)\sum_{i=1}^{n}\frac{x_i^2 e^{-(\lambda x_i)^2}}{1-e^{-(\lambda x_i)^2}} = 0 \qquad (2.2)$$

$$\frac{\partial logL}{\partial k} = 0 \Rightarrow \frac{n}{k} + \sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right) = 0 \qquad (2.3)$$

These equations show that maximum likelihood estimator of λ is an iterative solution involving *k* and maximum likelihood estimator of *k* is a closed form expression involving λ given as

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-(\lambda x_i)^2}\right)} \qquad (2.4)$$

Since λ is a scale parameter, in view of the scale invariant nature of the Burr type X distribution, maximum likelihood estimation of *k* can be taken as the following expression in a standard model.

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n} \ln\left(1-e^{-z_i^2}\right)} \qquad (2.5)$$

where $\lambda x_i = z_i$.

The elements of the information matrix and hence those of the asymptotic dispersion matrix require the following second order partial derivatives.

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda^2} = \frac{-2}{\lambda^2} - 2x^2 + \frac{2x^2(k-1)e^{-(\lambda x)^2}(1-2\lambda^2 x^2 - e^{-(\lambda x)^2})}{[1-e^{-(\lambda x)^2}]^2} \qquad (2.6)$$

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial k^2} = \frac{-1}{k^2} \qquad (2.7)$$

$$\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k} = \frac{2x^2 \lambda e^{-(\lambda x)^2}}{1-e^{-(\lambda x)^2}} \qquad (2.8)$$

The information matrix is given by

$$\begin{bmatrix} -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda^2}\right] & -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k}\right] \\ -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial \lambda\, \partial k}\right] & -nE\left[\frac{\partial^2 \log f(x;\lambda,k)}{\partial k^2}\right] \end{bmatrix}$$

As the integrals involved in these mathematical expectations are not analytically tractable we have evaluated them using 11- point Gauss-Laguerre quadrature formula (Rao *et al.,* 1966) [7] for selected values of *k* in a standard density (λ=1) and are given in Table - 2.1. The elements of the corresponding asymptotic dispersion matrix for selected values of *k* are given in Table – 2.2.

Table 2.1: Elements to get Information Matrix at *k*=2 and 3

k	2			3		
Matrix	I_{11}	I_{12}	I_{22}	I_{11}	I_{12}	I_{22}
	8.84668	-0.989225	0.25	10.9596	-0.68296	0.1111

Table 2.2: Elements of Asymptotic Dispersion Matrix of MLEs of λ & *k*

n \ ***k***	2			3		
	σ_{11}	σ_{12}	σ_{22}	σ_{11}	σ_{12}	σ_{22}
5	0.040548	0.160445	1.434864	0.029579	0.181811	2.917553
10	0.020274	0.080222	0.717432	0.014789	0.090905	1.458777
15	0.013516	0.053482	0.478288	0.00986	0.060604	0.972518
20	0.010137	0.040111	0.358716	0.007395	0.045453	0.729388
25	0.00811	0.032089	0.286973	0.005916	0.036362	0.583511

3. Two Step Estimation Procedure of Burr Type X Distribution

In this Section we discuss the point estimation of both the parameters λ and k. We notice that the log likelihood equations to get the MLEs of λ and k as presented in Section – 2 yield an analytical expression for the MLE of k whereas an iterative solution for λ. We suggest different methods of estimation for λ and k based on the same sample in order to get their estimates finally on the basis of a given sample. Such a procedure is called Two Step Estimation as done by Engeman and Keefe (1985) [2] for the Weibull distribution, Srinivasa Rao and Kantam (2012) [8] for log logistic distribution and the references there in. In the present situation we estimate λ as described below.

Consider the CDF of the two parameter Burr type X given by

$$F(x) = \left[1-e^{-(\lambda x)^2}\right]^k \qquad (3.1)$$

Let x_1, x_2 be two observations from this CDF with $x_1 < x_2$. Let $p_1 and\ p_2$ be two real numbers between 0 and 1such that, $p_1 < p_2$. Consider the two following equations

$$\left.\left[1-e^{-(\lambda x_1)^2}\right]^k = p_1\ ;\ \left[1-e^{-(\lambda x_2)^2}\right]^k = p_2 \right\} \qquad (3.2)$$

Taking logarithms and dividing we get

$$\frac{Ln\left[1-e^{-(\lambda x_1)^2}\right]}{Ln\left[1-e^{-(\lambda x_2)^2}\right]} = \frac{Ln(p_1)}{Ln(p_2)} \qquad (3.3)$$

Since $0 < e^{-(\lambda x)^2} < 1$, neglecting higher powers of $e^{-(\lambda x)^2}$ we suggest

$$Ln\left[1-e^{-(\lambda x)^2}\right] \cong e^{-(\lambda x)^2} \qquad (3.4)$$

Substituting this approximation in (3.3) and on simplification we get

$$\lambda = \sqrt{\frac{Ln\left[\frac{Ln(p_1)}{Ln(p_2)}\right]}{x_2^2 - x_1^2}} \qquad (3.5)$$

Thus with the suggested approximation (3.4), we see that two ordered observations are adequate to get an estimate of λ in a closed form as given in (3.5), wherein p_1, p_2 are generally taken as $\frac{1}{3}, \frac{2}{3}$ (in general $p_i = \frac{i}{n+1}$).

If we have a random sample of size more than 2 ($n \geq 3$) we propose the following procedure in order to use all the observations to get $\hat{\lambda}$ as given in equation (3.5). We order the sample of *n* observations in ascending manner and divide it into two halves – lower half and upper half of $\frac{n}{2}$ observations each, deleting the sample median in case the sample size is odd. We consider the arithmetic means of observations in lower half and upper half as x_1, x_2 respectively for use in the formula (3.5). This procedure uses all the sample observations but one to estimate λ if the sample size is greater than are equal to 4. We call this way of estimating λ as the first step of estimation.

The estimate of λ so obtained in the first step shall then be used in the expression for the exact MLE of k given by

$$\hat{k} = \frac{-n}{\sum_{i=1}^{n} Ln(1-e^{-(\lambda x_i)^2})}. \quad (3.6)$$

When both the parameters are unknown we suggest to estimate λ in the first step of estimation by dividing the ordered sample as lower and upper halves to make use of the formula (3.5). As a matter of academic interest, we suggest to use the arithmetic means $(\mathrm{AM_l}, \mathrm{AM_u})$, geometric means $(\mathrm{GM_l}, \mathrm{GM_u})$ and harmonic means $(\mathrm{HM_l}, \mathrm{HM_u})$ of the two halves of the divided ordered sample in succession in the place of x_1, x_2 of equation(3.5).

Next, in the second step of estimation, the parameter k is estimated by the exact ML method given by the formula (3.6) together with the estimate of λ obtained in the first step of estimation. We name this entire procedure as two step estimation - a hybrid of an adhoc method and an exact ML method. The performance of such a two-step procedure is studied in the following lines through Monte – Carlo simulation by computing the empirical variance covariance matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, k=2, 3, where $\hat{\lambda}$ is given by (3.5) and $\hat{k}$ is given by (3.6) over 10,000 runs. Results are presented in Tables 3.1 through 3.6.

For the sake of comparison with the asymptotic dispersion matrix of MLEs of λ, k, we present those elements also in Tables 3.7 and 3.8 for k= 2, 3. The trace, the determinant of the dispersion matrices called the generalised variances in the multi parameter case are also given in the respective tables. The ratios of the determinant of the asymptotic dispersion matrix from Tables 3.7 and 3.8 to the corresponding value of the determinant of the empirical dispersion matrix of the two step estimators for n=5(5)25, k= 2, 3 are given in Table – 3.9 naming them as Asymptotic Relative Efficiencies (ARE) of the two step estimation with respect to determinant of dispersion matrix as a generalised variance. Similarly, the ratios of trace of the asymptotic dispersion matrix to the corresponding trace of empirical dispersion matrix of the two step estimators are also given in Table – 3.9 as another measure of ARE with respect to another generalised variance. We see from Table – 3.9 that the suggested two step estimation is performing very encouragingly with the ARE more than one in most of the cases.

Table 3.1: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, at k=2 using A.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda}, \hat{k})$	**Determinant of the Matrix**	**Trace of the Matrix**
5	0.05055	3.730416	0.311044	0.091824	3.780966
10	0.020419	0.39924	0.088655	0.000292	0.419659
15	0.011113	0.18422	0.044888	3.23E-05	0.195333
20	0.00886	0.140375	0.03507	1.38E-05	0.149235
25	0.00643	0.101515	0.025387	8.24E-06	0.107945

Table 3.2: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, at k=3 using A.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda}, \hat{k})$	**Determinant of the Matrix**	**Trace of the Matrix**
5	0.046699	4.326304	0.417914	0.027382	4.373003
10	0.017762	0.920913	0.124873	0.000764	0.938675
15	0.009864	0.384948	0.060987	7.77E-05	0.394812
20	0.007331	0.287159	0.045482	3.66E-05	0.29449
25	0.005496	0.195254	0.032597	1.06E-05	0.20075

Table 3.3: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, at k=2 using G.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda}, \hat{k})$	**Determinant of the Matrix**	**Trace of the Matrix**
5	0.051894	2.522506	0.303581	0.038742	2.5744
10	0.019667	0.411583	0.087936	0.000362	0.43125
15	0.011554	0.187109	0.046195	2.79E-05	0.198663
20	0.008636	0.139725	0.034567	1.18E-05	0.148361
25	0.006321	0.100421	0.025048	7.36E-06	0.106742

Table 3.4: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, at k=3 using G.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda}, \hat{k})$	**Determinant of the Matrix**	**Trace of the Matrix**
5	0.045026	4.213351	0.396786	0.032271	4.258377
10	0.016873	0.923231	0.121203	0.000888	0.940104
15	0.009454	0.387436	0.059963	6.73E-05	0.39689
20	0.007368	0.286587	0.045558	3.6E-05	0.293955
25	0.00557	0.190946	0.032445	1.09E-05	0.196516

Table 3.5: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for n=5(5)25, at k=2 using H.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda}, \hat{k})$	**Determinant of the Matrix**	**Trace of the Matrix**
5	0.051899	2.11859	0.293146	0.024018	2.170489
10	0.021184	0.412044	0.092278	0.000214	0.433228
15	0.011424	0.195767	0.046908	3.61E-05	0.207191

20	0.008821	0.146911	0.035675	2.32E-05	0.155732
25	0.006602	0.09909	0.0255	3.94E-06	0.105692

Table 3.6: Empirical Dispersion Matrix of $\hat{\lambda}, \hat{k}$ for *n*=5(5)25, at *k*=3 using H.M.

n	$V(\hat{\lambda})$	$V(\hat{k})$	$Cov(\hat{\lambda},\hat{k})$	Determinant of the Matrix	Trace of the Matrix
5	0.044031	6.154925	0.433282	0.083274	6.198956
10	0.017684	0.873569	0.122188	0.000518	0.891253
15	0.009821	0.39546	0.061372	0.000117	0.405281
20	0.007642	0.299302	0.047438	3.69E-05	0.306944
25	0.005445	0.205883	0.033295	1.25E-05	0.211328

Table 3.7: Asymptotic Dispersion Matrix of MLEs of λ & *k*; Generalised Variances at *k*=2

n	σ_{11}	σ_{22}	σ_{12}	Determinant of the Matrix	Trace of the Matrix
5	0.040548	1.434864	0.160445	0.032438	1.475412
10	0.020274	0.717432	0.080222	0.00811	0.737706
15	0.013516	0.478288	0.053482	0.003604	0.491804
20	0.010137	0.358716	0.040111	0.002027	0.368853
25	0.00811	0.286973	0.032089	0.001298	0.295083

Table 3.8: Asymptotic Dispersion Matrix of MLEs of λ & *k*; Generalised Variances at *k*=3

n	σ_{11}	σ_{22}	σ_{12}	Determinant of the Matrix	Trace of the Matrix
5	0.029579	2.917553	0.181811	0.053243	2.947132
10	0.014789	1.458777	0.090905	0.01331	1.473566
15	0.00986	0.972518	0.060604	0.005916	0.982378
20	0.007395	0.729388	0.045453	0.003328	0.736783
25	0.005916	0.583511	0.036362	0.00213	0.589427

Table 3.9: Asymptotic Relative Efficiencies of Two Step Estimation

n	$k=2$						$k=3$					
	Using A.M.		Using G.M.		Using H.M.		Using A.M.		Using G.M.		Using H.M.	
	Trace	Determinant	Trace	Determinant	Trace	Determinant	Trace	Determinant	Trace	Determinant	Trace	Determinant
5	0.390221	0.353263	0.573109	0.837283	0.67976	1.35057	0.673938	1.944453	0.692079	1.649871	0.475424	0.639371
10	1.75787	27.77397	1.710623	22.40331	1.702812	37.8972	1.569836	17.42147	1.56745	14.98874	1.653364	25.69498
15	2.517772	111.5789	2.475569	129.1756	2.373675	99.8338	2.488217	76.139	2.47519	87.9049	2.423943	50.5641
20	2.471625	146.8841	2.486186	171.7797	2.368511	87.37069	2.501895	90.92896	2.506448	92.44444	2.400382	90.1897
25	2.733642	157.5243	2.764451	176.3587	2.791914	329.4416	2.936125	200.9434	2.999384	195.4128	2.789157	170.4

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