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A discussion on real number system

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Abstract
The real number system is one of the most elegant mathematical systems. The structure of real number system is a concrete example for the abstract mathematical system. Real number system is the foundation of the area of mathematics known as real analysis.

Keywords: real number, rational number, irrational number, terminating decimal, non-terminating decimal

1. Introduction

Number is one of the basic concepts of mathematics. Its origin dates back to ancient time. Today many kinds of numbers are known of all the different kinds of numbers, they were not discovered or created simultaneously. During the early days of human civilization primitive man created or invented the counting numbers in order to concretize their need and notion, which today known as natural number. The number 0 (zero) was introduced in the collection of natural numbers only a few hundred years ago by Hindu mathematicians. Negative numbers appeared very late in the history of numbers. Next came the fractional or rational numbers. It is believed that the fractions or rational numbers came much earlier than the negative numbers. The notion of fractions was motivated in a very natural way even in early days of human civilization. The subsequent introduction of the mysterious objects called irrational numbers did appear as early as the time of Pythagoras, no real advance towards the construction of a rigorous theory of irrational numbers took place till the time of Weierstrass (1815-1879) and Dadekind (1813-1916) [1]. The history of systematic development of real numbers, however, does not go back more than a century. It was only in the last quarter of the 19^{th} century that three proper theories of the structure of real numbers were separately proposed by three German mathematicians, K. Weierstrass, R. Dadekind and G. Cantor [2].

The inadequacy of rational numbers was known to ancient Greek Mathematicians, a satisfactory theory of real numbers was not available until late in the nineteenth century. During the 2^{nd} half of the nineteenth century, three different theories were propounded by three German mathematicians – Karl Weierstrass (1815 – 1997), Richard Dedekind (1831 – 1961) and Georg Cantor (1845 – 1918) [3].

The collection of rational numbers, consisting of the positive integers, zero, negative integers and common fractions together with the collection of irrational numbers constitute what is known as the set of real numbers.

2. Characteristics of the Real Numbers

The Properties of real number system fall into three categories: Algebraic properties, order properties and completeness.

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[1] R.M. Shrestha – Fundamentals of Mathematical Analysis "Revised edition – 2068, page 66, Sukunda Pustakk Bhawan, Kathmandu.

[2] S.L. Gupta, Nisha Rani – "Fundamental Real Analysis", Third revised edition, 1993, page 11. Vikas Publishing House Pvt. Ltd., New Delhi.

[3] Prof. SM. Maskey – "Principles of Real Analysis", Second edition – 2007, page-37, Ratna Pustak Bhandar, Kathmandu.

A. The algebraic properties say that the real number can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic.

B. The order properties of real number are:
If a, b, c $\in \Re$, then

i. Either $a < b$ or $a = b$ or $a > b$.
ii. $a < b \Rightarrow a + c < b + c$ or $a - c < b - c$
iii. $a < b$ and $c > 0 \Rightarrow ac < bc$ or, $a < b$ and $c < 0 \Rightarrow ac > bc$
iv. $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$
v. For $a < b$ there exist $c \in \Re$, such that $a < c < b$.
vi. C. The completeness properties of real numbers are:
vii. Every non – empty sets S of real numbers which is bounded above has a least upper bound.
viii. Every non – empty sets S of real numbers which is bounded below has a greatest lower bound.
ix. Let S be a non – empty bounded subset of $\Re$. Then
x. For every $\in > 0$, $\exists$ some real number $x \in S$ such that $x >$ sup. S - $\in$
xi. For every $\in > 0$, $\exists$ some real number $x \in S$ such that $x <$ Inf. S + $\in$
xii. Let A, B $\subseteq \Re$, Let $C = \{x + y: x \in A, y \in B\}$ then
xiii. Sup.A + Sup.B = Sup.C and Inf A + Inf B = Inf.C
xiv. Let A, B $\subseteq \Re$ such that $a \leq b$ for every $a \in A$ and $b \in B$. If B has a supremum, then A has a supremum and sup. A $\leq$ Sup. B.
xv. Again, if A has a supremum and B has an infimum then
xvi. Sup. A $\leq$ Inf. B
xvii. If $c > 0$, $b \in \Re$ then there exists $n \in \mathcal{N}$ such that $nc > b$.

2.1 Rational and Irrational Numbers

Quotients of integers $\frac{a}{b}$ (where $b \neq 0$) are called rational numbers, which is denoted by Q, contains Z as a subset. The real numbers that are not rational are called irrational. $\Re$–Q stands for the set of irrational numbers.

2.1.1 Characteristics of Rational and Irrational Numbers

The set of natural numbers are ordered. We can easily see that the set of integers and the set of rational numbers are also ordered. The set of rational numbers have predecessors or successors but not immediate predecessor and immediate successor. In the case of fractional numbers, one cannot speak of the 'immediate next' number. Between any two rational numbers there are infinitely many rational numbers, which implies that if we are given a certain rational number we cannot speak of the next largest rational numbers. This property of rational numbers is described by saying that the set of rational number is dense. That is the rational number system has certain gaps, in spite of the fact that between any two rational numbers we can insert other rational indefinitely. A rigorous construction of the irrational numbers is rather a difficult task at this stage. Other characters are:

i. If x and y are any rational numbers with $x < y$, then there exists a rational number r (in fact infinitely many) such that: $x < r < y$. [Known as Rational density Theorem]
ii. If x and y are real numbers with $x < y$, then there exists an irrational number t such that $x < t < y$. [Known as irrational Density Theorem]

$$\left[\begin{array}{c} \text{let } x, y \in \Re, r \in Q \Rightarrow \frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}} \in \Re \\ \text{So, } \frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \Rightarrow x < r\sqrt{2} < y \\ \text{where } t = r\sqrt{2} \text{ is an irrational number.} \end{array}\right]$$

iii. The set of rational numbers Q is not complete.
iv. The set of rational number is countable.
v. If n is a positive integer which is not a perfect square, then $\sqrt{n}$ irrational.
vi. Every Transcendental number is irrational.
vii. The number of the form $n^{1/m}$ are irrational unless n is the m^{th} power of an integer.
viii. The number of the form $\log_n m$ are irrational if m and n are integers.
ix. The number π (which is the ratio of the circumferences of the circle to its diameter) is irrational.
x. The Eular's number 'e' is a famous irrational number.

2.2 The Real Number as Decimal

The set of real numbers classified into the set of rational and irrational numbers. Any rational number can be expressed as a repeating or a terminating decimal. The set of irrational numbers being the complement of the set of rational numbers in the set of real number, the irrational numbers also can be expressed in decimals. A number represented by a non-repeating, non-terminating decimal is an irrational number i.e., irrational number in decimal goes on forever without repeating. Hence, the set $\Re$ of real number is the set of all terminating, repeating, and non-terminating non repeating decimals.

When using our calculator to determine if a decimal number is irrational. The calculator may not be displaying enough digits to show us the repeating and non-repeating decimals. When we use calculator, we get various types of rational and irrational numbers as follows:

$\sqrt{3}$ = 1.7320508075688772935274463415059………………….

π = 3.14159265358979323846264338327950…………… (The popular approximation of π is $\frac{22}{7}$ but not accurate.)

e = 2.71828182845904523536028747135270…………………. (Which is known as Eular's number)

$\sqrt{2}$ =1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727………[4]

Notice that there is no pattern among the digits, and there no repetition even of groups of digits. Does this mean that all rational fractions will have a period of digits? Let's inspect a few common fractions.

$\frac{1}{7}$ = $0.142857\underline{142857}142857\underline{142857}$………, which can be written as: 0.142857 (a six-digit period)

2.2.1 An Interesting Rational Number in Decimal

Suppose we consider the fraction $\frac{1}{109}$

$\frac{1}{109}$ = 0.009174311926605504587155963302752293577981651376146788990825688073394495142844036697247706422018348 6……[5]

Here we have calculated its value to 100 places and no period appears. Does this mean that the fraction is irrational? This would destroy our nice definition above. We can try to calculate the value a bit more accurately, that is, say, to another 12 places.

$\frac{1}{109}$ = 0.00917431192660550458715596330275229357798165137614678899082568807339449541284403669724770642201834862385321100 91 [6]

Suddenly it looks as though a pattern may be appearing; the 0091 also began the period.

We carry out our calculation further to 220 places and notice that a 108-digit period emerges.

$\frac{1}{109}$ = 0.0091743119266055045871559633027522935779816513761467889908256880733944954128440366972477064220183486238532110091 74

3119266055045871559633027522935779816513761467889908256880733944954128440366972477064220183486238532110091 74
[7]

If we carry out the calculation to 330 places, the pattern becomes clearer.

$\frac{1}{109}$ = 0.009174311926605504587155963302752293577981651376146788990825688073394495412844036697247706422018348 6238

532110091743119266055045871559633027522935779816513761467889908256880733944954128440366972477064220183 48
623853211009174311926605504587155963302752293577981651376146788990825688073394495412844036697247706422 01
8348623853211009174 [8]

We might be able to conclude that a common fraction results in a decimal equivalent that has a repeating period of digits. Some common ones we already are familiar with, such as:

$\frac{1}{3}$ =.33333333$\overline{3}$

$\frac{1}{13}$ = 0.076923076923076923$\overline{076923}$0

3. Conclusion

The set of ℜ acts as the universal set U in respect of sets consisting of only real numbers. The properties of the real number system fall into three categories they are algebraic, order and completeness. Completeness property of the real number system is deeper and harder to define precisely roughly speaking, there are enough real numbers to complete the real number line, in the

[4] Albrechit, Alfred S., "Brain storming in mathematics" - 2000 (Third Revised edition) Narosa publishing House, New Delhi, (page 103 - 120)
[5], [6], [7], [8] Albrechit, Alfred S., "Brain storming in mathematics" - 2000 (Third Revised edition) Narosa publishing House, New Delhi, (page 103 - 120)

sense that there are no 'holes' or 'gaps' in the real line. The rational numbers are precisely the real numbers with decimal expansions that are either terminating or repeating. The set of real numbers has all the algebraic and order properties of the real numbers but lacks the completeness properties. We saw that a common fraction will result in a repeating decimal, sometimes with a very long period and sometimes with a very short period. It would appear, from the rather unbelievable evidence so far, that a fraction results in a repeating decimal and an irrational number does not. Yet this does not prove that an irrational number cannot be expressed as a fraction. The real number system is one of the most elegant mathematical system. The structure of real number system is a concrete example for the abstract mathematical system. Real number system is the foundation of the area of mathematics known as real analysis.

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