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Mathematical modelling of physical and hydraulic system with engineering

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Abstract
In Case of System Mathematical Model plays an important role to give response. The process of developing mathematical Model is known as Mathematical Modelling. Modelling is the process of writing a differential equation to describe a physical situation and this paper explains different kinds of System such as electrical, mechanical, and hydraulic. In Accordance of it examples of Mechanical, Electrical and Hydraulic system are represented by mathematical model; in different types of Mathematical model i.e. Mechanical System by Differential Equation Model, Electrical system by State-Space Model and Hydraulic System by Transfer Function Model.

Keywords: Mathematical Modelling, Electrical, Mechanical and Hydraulic Systems and their Behaviour.

Introduction
For the analysis and design of control systems, we need to formulate a mathematical description of the system. The process of obtaining the desired mathematical description of the system is known as "Modelling". The basic models of dynamic physical systems are differential equations obtained by application of the appropriate laws of nature. These equations may be linear or nonlinear depending on the phenomena being modelled. The differential equations are inconvenient for the analysis and design manipulations and so the use of Laplace Transformation which converts the differential equations into algebraic equations is made use of. The algebraic equations may be put in transfer function form, and the system modelled graphically as a transfer function block diagram. Alternatively, a signal flow graph may be used. The differential equations are inconvenient for the analysis and design manipulations and so the use of Laplace Transformation which converts the differential equations into algebraic equations is made use of. The algebraic equations may be put in transfer function form, and the system modelled graphically as a transfer function block diagram. Alternatively, a signal flow graph may be used. This paper concerned with differential equations, transfer functions, block diagrams, signal flow graphs, etc., of different physical systems namely, mechanical, electrical, hydraulic, pneumatic and thermal systems. Analysis of a dynamic system requires the ability to predict its performance. This ability and the precision of the results depend on how well the characteristics of each component can be expressed mathematically. One of the most important tasks in the analysis and design of control systems is mathematical modelling of the systems. The two most common methods are the transfer function approach and the state equation approach. The transfer function method is valid only for linear time-invariant systems, whereas the state equations are first-order to use transfer functions and linear state equations the system must first be linearized, or its range of operation must be confined to a linear range. Although the analysis and design of linear control systems have been well developed, their counterparts for nonlinear systems are usually quite complex. Therefore, the control systems engineer often has the task of determining not only how to accurately describe a system mathematically, but also, more important, how to make proper assumptions and approximations, whenever necessary, so that the system may be adequately characterized by a linear mathematical model. The System is used to describe a combination of component which may be physical or may not. Mathematical model describes the system in terms of mathematical concept.

The process of developing mathematical Model is known as Mathematical Modelling. Modelling is the process of writing a differential equation to describe a physical situation. The basis for mathematical model is provided by the fundamental physical laws that govern the behaviour of system. It uses laws like Kirchhoff's law for electrical system, Newton's law for mechanical system. Modelling of any system can help us to study effect of different of component and to make Prediction about Behaviour. Modelling can be divided into two parts i.e. First Principle Model and empirical model given in figure 1.

- First principle model that seeks to calculate a physical quantity starting directly from established laws of physics without making any assumptions. Example-Electronic structure of atoms
- An Empirical modelling refers to any kind of modelling based on empirical observations rather than mathematically describable relationships of the system modelled.

Example-Tank System

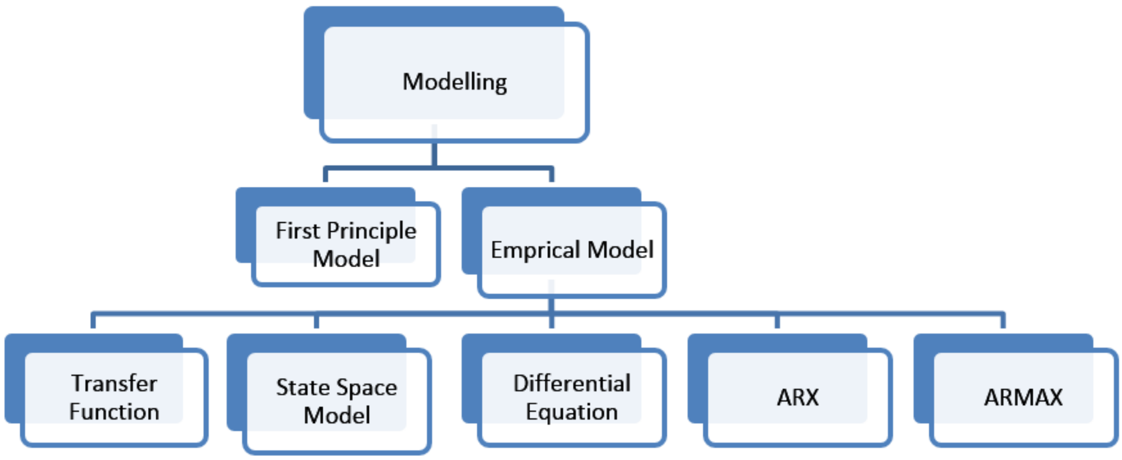


Fig 1: Gives Types and Subtypes of Modelling

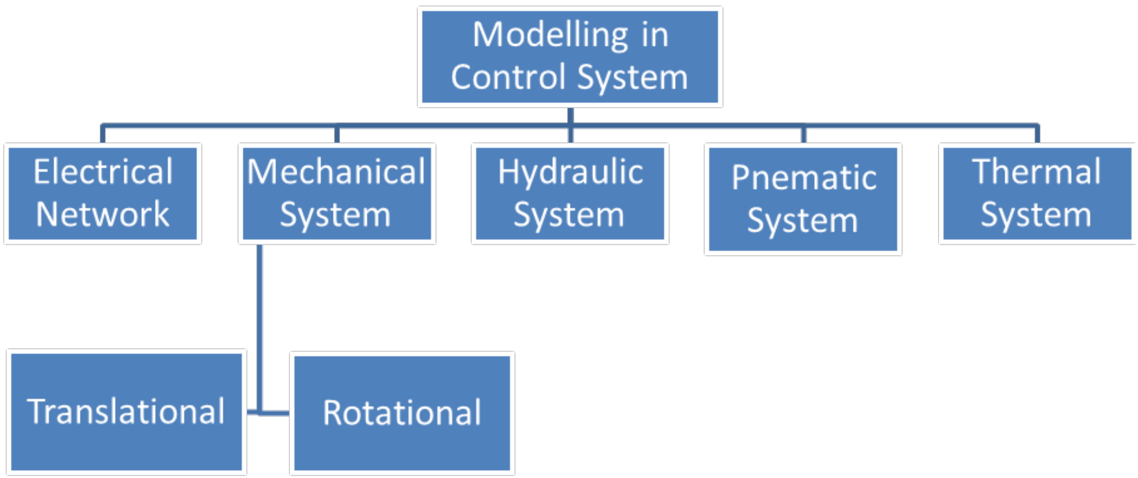


Fig 2: Types of control system

Mechanical system elements

The Electrical System is made with element like Resistor, Capacitor and Inductor here, We can use mathematical modelling to check the behaviour of any electrical system. Most feedback control systems contain mechanical as well as electrical components. From a mathematical viewpoint, the descriptions of electrical and mechanical elements are analogous. In fact, we can show that given an electrical device, there is usually an analogous mechanical counterpart, and vice versa. The analogy, of course, is a mathematical one; that is, two systems are analogous to each other if they are described mathematically by similar equations.

Translational Motion

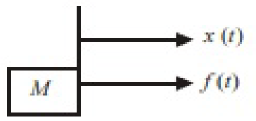
Translational motion takes place along a straight line and the variables involved in describing a straight-line motion are displacement, velocity and acceleration. Newton's law of motion governs the linear motion. According to this law, the product of mass and acceleration is equal to the algebraic sum of forces acting on it. Newton's law of motion states that the algebraic sum of forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. The law can be expressed as

$$\Sigma \text{ forces} = Ma \quad (1)$$

where M denotes the mass and a is the acceleration in the direction considered.

Mass: The function of mass in linear motion is to store kinetic energy. Mass cannot store potential energy. Suppose a force is applied to mass M as shown in Fig. 2.1, the mass starts moving in x direction as shown. For the time being, we will assume other forces such as friction, etc. to be zero. Hence, according to Newton's law,

$$M\frac{d^2x}{dt^2} = f(t)$$



(2)

Fig (2.1)

Resistor: Resistor is a passive two-terminal electrical component that implements electrical resistance as a circuit element. The current through a resistor is in direct proportion to the voltage across the resistor's terminals. This relationship is represented by Ohm's law:

I= V/I

Where I is the current through the conductor in units of amperes, V is the potential difference measured across the conductor in units of volts, and R is the resistance of the conductor in units of ohms. The ratio of the voltage applied across a resistor's terminals to the intensity of current in the circuit is called its resistance, and this can be assumed to be a constant (independent of the voltage) for ordinary resistors working within their ratings.

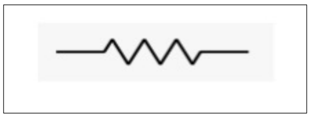


Fig 3: Resistor

Capacitor: Capacitor is passive electrical component used to store energy electrostatically in an electric field. The forms of practical capacitors vary widely, but all contain at least two electrical conductors separated by a dielectric; for example, one common construction consists of metal foils separated by a thin layer of insulating film. When there is a potential difference across the conductors, an electric field develops across the dielectric, causing positive charge to collect on one plate and negative charge on the other plate. Energy is stored in the electrostatic field.

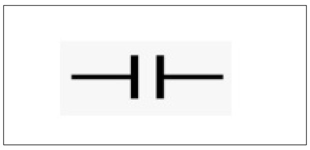


Fig 4: Capacitor

Voltage across Resistor given by

$$\frac{1}{C}\int I_C\, dt$$

Inductor: An inductor is characterized by its inductance, the ratio of the voltage to the rate of change of current, which has units of Henry (H). Many inductors have a magnetic core made of iron or ferrite inside the coil, which serves to increase the magnetic field and thus the inductance.

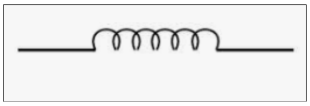
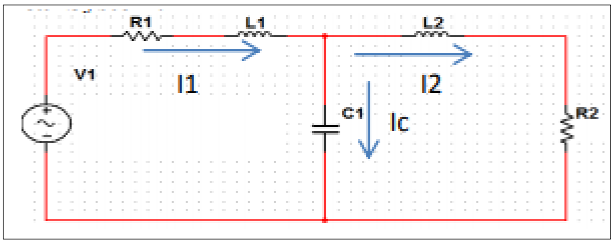


Fig 5: Inductor

Current I_L through inductor is given by

$$\frac{1}{L}\int E_L\, dt$$

Example of Electrical System:



By Applying KCL $I_C = I_1 - I_2$

$$C\frac{dV}{dt} = I_1 - I_2$$

By Applying KVL at inductor 1

$$V_i = R_1 I_1 + L_1\frac{dI_1}{dt} + V_c$$

$$\frac{dI_1}{dt} = \frac{1}{L_1}(V - R_1 I_1 - V_c)$$

By Applying KVL At Inductor 2

$$V_c = L_2\frac{dI_2}{dt} + R_2 I_2$$

$$\frac{dI_2}{dt} = \frac{1}{L_2}(V_c - R_2 I_2)$$

State Space Representation is given by

$$\begin{bmatrix}\dot{V}_c\\ \dot{I}_1\\ \dot{I}_2\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{C1} & \frac{1}{C1}\\ -\frac{1}{L1} & -\frac{R1}{C1} & 0\\ \frac{1}{L2} & 0 & \frac{R2}{L2}\end{bmatrix}\begin{bmatrix}V_c\\ I_1\\ I_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{L1}\\ 0\end{bmatrix}[V_i]$$

$$V_o = [0\ 0\ R2]\begin{bmatrix}V_c\\ I_1\\ I_2\end{bmatrix}$$

Mechanical System:
Elements of Mechanical System

1. Mass
- A Force applied to the mass produces an acceleration of the mass.
- The reaction force fm is equal to the product of mass and acceleration and is opposite in direction to the applied force in term of displacement y, a velocity v, and acceleration a, the force equation is

$$F = Ma = M\dot{v} = M\ddot{x}$$

Fig 6: Mass

2. Spring:
- The reaction force F on each end of the spring is the same and is equal to the product of stiffness k and the amount of deformation of the spring.
- End C has a position Yc and end D has a position Yd measured from the respective equilibrium positions. The force equation, in accordance with the Hok's law is

F = k (Yc -Yd)

- If the end D is stationary, then Yd = 0 and the above equation reduces to

F = kYc

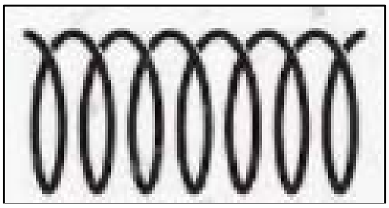


Fig 7: Spring

3. Damper
- The reaction damping force FB is approximated by the product of damping B and the relative velocity of the two ends of the dashpot.
- The direction of this force depend on the relative magnitude and direction of the velocity Dye and Dyf.

$$F_b = B(V_e - V_f) = B(\dot{D}_{y_e} - \dot{D}_{y_f})$$

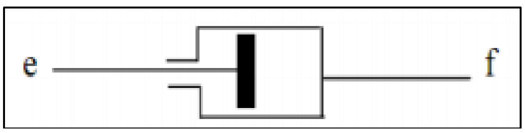


Fig 8: Damper

Mechanical System Example:
It is required to set up equation for system on application of force F(t) to mass m. The resulting displacement of mass being x

$$\text{Inirtial force} = M\frac{d^2x}{dt^2}$$

$$\text{Viscous Force} = f\frac{dx}{dt}$$

$$\text{Spring Restoring Force} = Kx$$

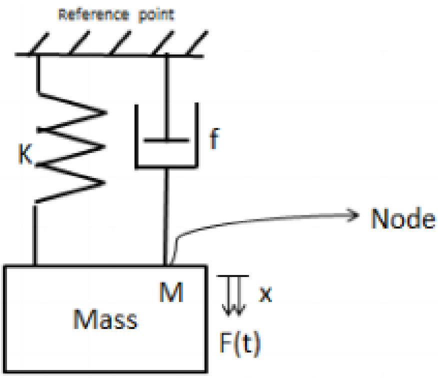


Fig 9: Example of Mechanical System

By Applying Nodal Analysis We Get

$$M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx = f(t)$$

By Assuming initial conditions are zero and taking Laplace Transforms We have

$$Ms^2X(s) + fsX(s) + kX(s) = F(s)$$

Transform Equation Can be given as

$$G(s) = \frac{1}{Ms^2 + fs + K}$$

Hydraulic System:
Elements of Hydraulic analogy

1. **Fluid Resistance:**
- A construction in the bore of the pipe which requires more pressure to pass the same amount of water. All pipes have some resistance to flow, just as all wires have some resistance to current.
- Laminar Flow

$$P = \frac{128lu}{\pi D^4}Q = RQ \qquad R = \frac{128lu}{\pi D^4}$$

For Turbulent Flow

$$P = \frac{8ktpl}{\pi^2 D^5}Q^2 = RQ \qquad R = \frac{8ktpl}{\pi^2 D^5}$$

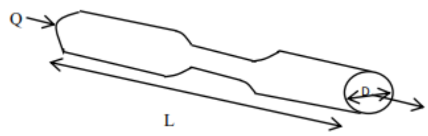


Fig 10: Flow of liquid in Pipe

l = length of pipe
D = Diameter of pipe
u = viscosity
Q = volumetric flow rate
Kt = constant
Þ = mass density

2. Fluid Capacitance:

- The other element used in modeling fluid system is capacitance.
- In case when fluid is stored it carries potential energy act like a capacitor.
- Rate of fluid storage in Tank

$$A\frac{dH}{dt} = \frac{A}{\rho g}\frac{dP}{dt} = C\frac{dP}{dt}$$

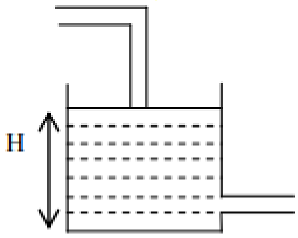


Fig 11: Fluid Tank

H=Height of liquid level
A=Tank cross sectional area
g=Gravitational acceleration
Þ=Mass density
• Capacitance:

$$C = \frac{A}{\rho g}$$

3. Fluid Inertance

- Inertial effect of fluid in pipe line is modelled as inertance.
- Change in Pressure of liquid in presence of inertance can be given as

$$\Delta P = L\frac{dQ}{dt}$$

- Inertance

$$L = \frac{\rho l}{A}$$

4. Example of Hydraulic System:

Rate of fluid storage in a tank =

$$A\frac{dH}{dt} = C\frac{dH}{dt}$$

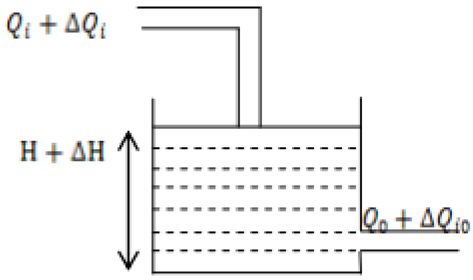


Fig 12: Fluid Tank

Under steady state condition

$$Q_i = Q_o$$

Let ΔQi be a small increase in liquid inflow rate from its steady state value this increased liquid inflow rate causes increase of head of liquid in tank by ΔH .Resulting increase of liquid outflow by

$$\Delta Q_o = \frac{\Delta H}{R}$$

Rate of liquid storage in tank = Rate of liquid Inflow- Rate of liquid Outflow

$$C\frac{d\Delta H}{dt} = \Delta Q_i - \Delta Q_o$$

$$C\frac{d\Delta H}{dt} = \Delta Q_i - \frac{\Delta H}{R}$$

$$RC\frac{d\Delta H}{R} + \Delta H = R(\Delta Q_i)$$

By Taking Laplace Transform

$$RC.sH(s) + H(s) = RQ_i(s)$$

Transfer Function:

$$\frac{H(s)}{Q_i(s)} = \frac{R}{RC.s + 1}$$

Transfer function

It has been shown already that the input and output of a linear system in general, is related by a linear or a set of linear differential equations. Such relationships are capable of completely describing the system behaviour in the presence of a particular input excitation and known initial conditions. Differential equation of Eq.(A) is seldom used in its original form for the analysis and design of control systems. To obtain the transfer function of the linear system that is represented by Eq.(A), we simply take the Laplace transform on both sides of the equation, and assume zero initial conditions. The result is

$$(s^n + a_n s^{n-1} + \ldots + a_2 s + a_1)C(s) = (b_{m+1}s^m + b_m s^{m-1} + \ldots + b_2 s + b_1)R(s) \quad \text{(A)}$$

The transfer function between r(t) and c(t) is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_{m+1}s^{m} + b_{m}s^{m-1} + \ldots + b_{2}s + b_{1}}{s^{n} + a_{n}s^{n-1} + \ldots + a_{2}s + a_{1}} \quad \textbf{(B)}$$

We can summarize the properties of the transfer function as follows:

1. Transfer function is defined only for a linear time-invariant system. It is meaningless for nonlinear systems.
2. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response. Alternately, the transfer function between a pair of input and output variables is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. When defining the transfer function, all initial conditions of the system are set to zero.
4. The transfer function is independent of the input of the system.
5. Transfer function is expressed only as a function of the complex variable s. It is not a function of the real variable, time, or any other variable that is used as the independent variable.

Conclusion

In Order to understand the behaviour of systems, Mathematical Models are needed. These are simplified representations of certain aspects of real system. Such a model is created using equations to describe the relationship between input and output of system and can then be used to enable prediction to be made of the behaviour of a system under specific condition.

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