A study of \( W_8 \)-curvature tensor in \( K \)-contact Riemannian manifold

PW Njori, Moindi SK and GP Pokhariyal

Abstract
In this paper, \( W_8 \)-curvature tensor is studied in \( K \)-contact Riemannian manifold. The semi-symmetric and symmetric properties with respect to the \( W_8 \)-curvature tensor are also studied.

Preliminaries
Let \((M, \phi, \xi, \eta, g)\) be \( n = (2m + 1) \)-dimensional almost contact Riemannian manifold consisting of a \((1,1)\) tensor field \( \phi \), a vector field \( \xi \), a 1-form \( \eta \) and a Riemannian metric \( g \).

\[
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0.
\]  

\((1.1)\)

(Pokhariyal) \[4\]

Keywords: \( W_8 \)-curvature tensor, \( W_8 \)-flat \( K \)-contact Riemannian manifold, \( W_8 \)-semi-symmetric and symmetric \( K \)-contact Riemannian manifold, \( W_8 \)-recurrent \( K \)-contact Riemannian manifold

1. Introduction

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(Y)\eta(X)
\]

\((1.2)\)

Where \( X, Y \) are arbitrary vector fields on \( M \). If moreover,

\[
g(X, \phi Y) = -g(\phi X, Y)
\]

\[
g(X, \nabla Y \xi) = -g(\nabla X \xi, Y) \quad \Leftrightarrow \quad \nabla X \xi = -\phi X
\]

\((1.3)\)

Then, \( M \) is a \( K \)-contact Riemannian manifold. Where \( \nabla \) denotes the Riemannian connection of \( g \).

In a \( K \)-contact manifold the following relations hold:

\[
\nabla X \xi = -\phi X
\]

\((1.4)\)

\[
S(X, \xi) = (n - 1)\eta(X)
\]

\((1.5)\)

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)\xi
\]

\((1.6)\)

The following statements are true about \( K \)-contact manifold. If for an almost contact manifold \( M^n \).
\[ \nabla_X \xi = -\phi X \quad \text{then } M^n \text{ is a K-contact manifold} \quad (1.7) \]
\[ g(X, \nabla_Y \xi) = -g(\nabla_X \xi, Y) \quad \text{then } M^n \text{ is a K-contact manifold} \quad (1.8) \]
\[ g(X, \phi Y) = -g(\phi X, Y) \quad \text{then } M^n \text{ is a K-contact manifold} \quad (1.9) \]
\[ \text{is both contact manifold and } \xi \text{ is a Killing vector, then } M^n \text{ is a K-contact manifold} \quad (1.10) \]

2. \( W_8 \) curvature tensor in K-contact Riemannian manifold

Pokhariyal [4] gave definition of \( W_8 \) curvature tensor as

\[
W_8(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} \left[ S(X, Y)Z - S(Y, Z)X \right]
\]

(2.1)

**Definition 2.1:** A K-contact Riemannian manifold \( M^n \) is said to be flat if the Riemannian curvature tensor vanishes identically, i.e. \( R(X, Y)Z = 0 \)

**Definition 2.2:** A K-contact Riemannian manifold \( M^n \) is said to be \( W_8 \)-flat if the \( W_8 \) curvature tensor vanishes identically, i.e. \( W_8(X, Y)Z = 0 \)

**Theorem 2.3:** A \( W_8 \)-flat K-contact Riemannian manifold is a flat manifold.

**Proof:** If \( W_8 \)-flat

If our hypothesis is true, then \( W_8 = 0 \) in

\[
W_8(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} \left[ S(X, Y)Z - S(Y, Z)X \right]
\]

Expanding (2.1) with respect to variable \( U \)

\[
W_8'(X, Y, Z, U) = R'(X, Y, Z, U) + \frac{1}{n-1} \left[ S(X, Y)g(Z, U) - S(Y, Z)g(X, U) \right]
\]

(2.2)

Therefore, if K-contact manifold \( M \) is \( W_8 \)-flat then, we have,

\[
R'(X, Y, Z, U) = \frac{1}{n-1} \left[ S(Y, Z)g(X, U) - S(X, Y)g(Z, U) \right]
\]

(2.3)

Where, \( S(X, Y) = Ric(X, Y) = (n-1)g(X, Y) \) Then, using \( S(X, Y) = (n-1)g(X, Y) \) in

\[
R'(X, Y, Z, U) = \frac{n-1}{n-1} \left[ g(Y, Z)g(X, U) - g(X, Y)g(Z, U) \right]
\]

(2.3)

\[
R'(X, Y, Z, U) = \left[ g(Y, Z)g(X, U) - g(X, Y)g(Z, U) \right]
\]

(2.4)

But, in K-contact manifold, we have

\[
R'(X, Y, Z, U) = \left[ g(Y, Z)g(X, U) - g(X, Z)g(Y, U) \right]
\]

From the computations, we get

\[ R'(X, Y, Z, U) = \left[ g(Y, Z)g(X, U) - g(X, Y)g(Z, U) \right] \]

Thus, for this to hold, we must have

\[ R'(X, Y, Z, U) = 0 \text{ since} \]

\[ R'(X, Y, Z, U) \neq \left[ g(Y, Z)g(X, U) - g(X, Y)g(Z, U) \right] \text{ by definition.} \quad (2.5) \]
This completes the theorem.

**Corollary 2.5:** A $W_8$ — flat K-contact manifold is neither Einstein nor $\eta - Einstein$ Manifold

3. $W_8$ – Semi-symmetric K-contact Riemannian manifold

De and Guha [5] gave the definition of semi-symmetric as

$$R(X, Y)R(Z, U)V = 0$$

(3.1)

**Definition 3.1:** A K-contact manifold is said to be $W_8$ — semi-symmetric if

$$R(X, Y)W_8(Z, U)V = 0$$

(3.2)

**Theorem 3.2:** $W_8$ – semi-symmetric K-contact manifold is a $W_8$ – flat manifold.

**Proof:** If the K-Contact manifold is a $W_8$ – semi-symmetric then $R(X, Y)W_8(Z, U)V = 0$

$$R(X, Y)W_8(Z, U)V = g(Y, W_8(Z, U)V)X - g(X, W_8(Z, U)V)Y = 0$$

$$\Rightarrow g(Y, W_8(Z, U)V)X - g(X, W_8(Z, U)V)Y = 0$$

$$\Rightarrow W_8'(Y, Z, U, V)X - W_8'(X, Z, U, V)Y = 0$$

(3.3)

$$\Rightarrow g(W_8'(Y, Z, U, V)X, \xi) - g(W_8'(X, Z, U, V)Y, \xi) = 0$$

$$\Rightarrow W_8'(Y, Z, U, V)\eta(X) - W_8'(X, Z, U, V)\eta(Y) = 0$$

Note, this is only possible if $W_8'(Y, Z, U, V) = 0$ and $W_8'(X, Z, U, V) = 0$ since $A(X) \neq 0$ and $A(Y) \neq 0$ and thus follows the theorem.

**Corollary 3.3:** A $W_8$ – semi-symmetric K-contact manifold is neither Einstein nor $\eta$-Einstein manifold.

4. $W_8$ – symmetric K-Contact Riemannian manifold

Chaki and Gupta [6] gave the definition of a conformally symmetric manifold as for which $\nabla U C = 0$ which is said to be conformally symmetric (where $C$ is conformal curvature tensor).

**Definition 4.1:** A K-contact Riemannian manifold $M$ is said to be $W_8$ – symmetric if

$$\nabla U W_8(X, Y)Z = 0$$

(4.1)

**Theorem 3.2:** $W_8$ – symmetric and a $W_8$ – flat K-contact Riemannian manifold is a flat-manifold.

**Proof:** If the K-contact space is a $W_8$ – symmetric and $W_8$ – semi-symmetric then it follows

$$0 = R(X, Y)W_8(Z, U)V - W_8'(R(X, Y)Z, U)V - W_8(Z, R(X, Y)U)V$$

$$- W_8'(Z, U)R(X, Y)V$$

(3.2)

Computing each of the above four terms separately yields

$$R(X, Y)W_8(Z, U)V = g(Y, W_8(Z, U)V)X - g(X, W_8(Z, U)V)Y$$


$$g(R(X, Y, W_8(Z, U)V, \xi) = g(W_8'(Y, Z, U, V)X, \xi) - W_8'(X, Z, U, V)Y, \xi)$$

$$= \eta(W_8'(Y, Z, U, V)X) - \eta(W_8'(X, Z, U, V)Y)$$

$$= W_8'(Y, Z, U, V)\eta(X) - W_8'(X, Z, U, V)\eta(Y)$$
Again,

\[
W_8(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1} \left[ S(R(X,Y)Z,U)V - S(U,V)R(X,Y)Z \right] 
\]

(3.4)

\[
= R(R(X,Y)Z,U)V + \frac{n-1}{n-1} \left[ g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z \right] 
\]

\[
= R(R(X,Y)Z,U)V + \left[ g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z \right] 
\]

(3.5)

\[
W'_8(R(X,Y)Z,U,V,\xi) = g(R'(X,Y,Z,U)V,\xi) - g(R'(X,Y,Z,V)U,\xi) 
\]

Next, we put together (3.3), (3.4), (3.5) and (3.6) to have

\[
W'_8(Y,Z,U,V)\eta(X) - W'_8(X,Z,U,V)\eta(Y) 
\]

(3.7)

\[
- \{ R'(X,Y,Z,U)\eta(V) - R'(X,Y,Z,V)\eta(U) 
\]

\[
+ R'(X,Y,U,Z)\eta(V) - g(Z,V)R'(X,Y,U,\xi) 
\]

\[
+ g(Z,U)R'(X,Y,V,\xi) - R'(X,Y,V,Z)\eta(U) \} = 0 
\]

Terms which are coefficients of \( \eta(V) \) and \( \eta(U) \) cancel out since \( R' \) is skew-symmetric with respect to the last two variables. Hence, (3.7) reduces to

\[
W'_8(Y,Z,U,V)\eta(X) - W'_8(X,Z,U,V)\eta(Y) + g(Z,V)R'(X,Y,U,\xi) 
\]

(3.8)

\[
- g(Z,U)R'(X,Y,V,\xi) = 0 
\]

but it is a \( W_8 \) – flat manifold, hence \( W'_8 = 0 \)

Therefore (3.8) reduces to

\[
(Z,V)R'(X,Y,U,\xi) - g(Z,U)R'(X,Y,V,\xi) = 0 
\]

(3.9)
But in (3.9)
\[ g(Z, U) \neq g(Z, V) \neq 0 \Rightarrow R' = 0 \]  
(3.10)

Thus, follows the theorem.

5. A \( W_8 \) – Recurrent K-contact Riemannian manifold.

**Definition 5.1:** A K-contact Riemannian manifold is said to be recurrent if
\[
(\nabla_U W_8)(X, Y)Z = B(U)W_8(X, Y)Z
\]
(5.1)

Where \( B \) is a non-zero 1-form.

**Theorem: 5.2:** A \( W_8 \)-recurrent and \( W_8 \)-flat manifold is a flat manifold.

**Proof:** We have
\[
(\nabla_U W_8)(X, Y)Z = B(U)W_8(X, Y)Z \quad \text{where} \quad B(U) \neq 0
\]
(5.2)

but, if \( W_8(X, Y)Z = 0 \)

Hence, (5.2) by definition becomes
\[
0 = R'(X, Y, Z, U) + \frac{1}{n-1} [S(X, Y)g(Z, U) - S(Y, Z)g(X, U)]
\]
(5.3)

\[
R'(X, Y, Z, U) = [g(Y, Z)g(X, U) - g(X, Y)g(Z, U)]
\]
(5.4)

But, for a K-contact manifold
\[
R'(X, Y, Z, U) = [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]
\]
(5.5)

So (5.4) can only be true if and only if
\[
R'(X, Y, Z, U) = 0
\]

And therefore, the theorem follows.

6. Reference
5. UC DC, Guha N. On coharmonically recurrent Sasakian manifold, Indian J. Math 1992, 34209-215