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A study of W_8-curvature tensor in K-contact riemannian manifold

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Abstract
In this paper, W_8 - curvature tensor is studied in K-contact Riemannian manifold. The semi-symmetric and symmetric and flatness properties with respect to the W_8 - curvature tensor are also studied.

Preliminaries
Let (M,ϕ,ξ,η,g) be $n=(2m+1)$ -dimensional almost contact Riemannian manifold consisting of a (1,1) tensor field ϕ , a vector field ξ , a 1-form η and a Riemannian metric g .

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi)=1,\ \phi\xi=0,\ \eta(\phi X)=0, \tag{1.1}$$

(Pokhariyal) [4]

Keywords: W_8 - curvature tensor, W_8- flat K-contact Riemannian manifold, W_8 -semi-symmetric and symmetric K-contact riemannian manifold. W_8 - recurrent K-contact riemannian manifold

1. Introduction

$$g(\phi X,\phi Y) = g(X,Y) - \eta(Y)\eta(X) \tag{1.2}$$

Where X, and Y are arbitrary vector fields on M
If moreover,

$$g(X,\phi Y) = -g(\phi X,Y)$$
$$g(X,\nabla_Y \xi) = -g(\nabla_X \xi, Y) \Leftrightarrow \nabla_X \xi = -\phi X \tag{1.3}$$

Then, M is a K-contact Riemannian manifold.
Where ∇ denotes the Riemannian connection of g.
In a K-contact manifold the following relations hold:

$$\nabla_X \xi = -\phi X \tag{1.4}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{1.5}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)\xi \tag{1.6}$$

The following statements are true about K-contact manifold. If for an almost contact manifold M^n,

- $\nabla_X \xi = -\phi X$ then M^n is a K-contact manifold (1.7)
- $g(X, \nabla_Y \xi) = -g(\nabla_X \xi, Y)$ then M^n is a K-contact manifold (1.8)
- $g(X, \phi Y) = -g(\phi X, Y)$ then M^n is a K-contact manifold (1.9)
- is both contact manifold and ξ is a Killing vector, then M^n is a K-contact manifold (1.10)

2. W_8 − curvature tensor in K-contact Riemannian manifold

Pokhariyal [4] gave definition of W_8 − curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right] \tag{2.1}$$

Definition 2.1: A K-contact Riemannian manifold M^n is said to be flat if the Riemannian curvature tensor vanishes identically, i.e. $R(X,Y)Z = 0$

Definition 2 .2: A K-contact Riemannian manifold M^n is said to be W_8 -flat if the W_8 − curvature tensor vanishes identically, i.e. $W_8(X,Y)Z = 0$

Theorem 2.3: A W_8 -flat K-contact Riemannian manifold is a flat manifold.

Proof; If W_8 -flat

If our hypothesis is true ,then $W_8 = 0$ in $W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right]$

Expanding (2.1) with respect to variable U

$$W_8'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}\left[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\right] \tag{2.2}$$

Therefore, if K-contact manifold M is W_8 − flat then, we have,

$$R'(X,Y,Z,U) = \frac{1}{n-1}\left[S(Y,Z)g(X,U) - S(X,Y)g(Z,U)\right] \tag{2.3}$$

Where, $S(X,Y) = Ric(X,Y) = (n-1)g(X,Y)$ Then, using $S(X,Y) = (n-1)g(X,Y)$ in

$$R'(X,Y,Z,U) = \frac{n-1}{n-1}\left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right] \tag{2.3}$$

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right] \tag{2.4}$$

But, in K-contact manifold, we have

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right]$$

From the computations, we get

$$\Rightarrow R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right]$$

Thus, for this to hold, we must have

$R'(X,Y,Z,U) = 0$ since

$R'(X,Y,Z,U) \neq \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right]$ by definition. (2.5)

This completes the theorem.

Corollary 2.5: A $W_8 -$ flat K-contact manifold is neither Einstein nor $\eta - Einstein$ Manifold

3. $W_8 -$ Semi-symmetric K-contact Riemannian manifold

De and Guha [5] gave the definition of semi-symmetric as

$$R(X,Y)R(Z,U)V = 0 \tag{3.1}$$

Definition 3.1: A K-contact manifold is said to be $W_8 -$ semi-symmetric if

$$R(X,Y)W_8(Z,U)V = 0 \tag{3.2}$$

Theorem 3.2: $W_8 -$ semi-symmetric K-contact manifold is a $W_8 -$ flat manifold.

Proof: If the K-Contact manifold is a $W_8 -$ semi-symmetric then $R(X,Y)W_8(Z,U)V = 0$

$$\begin{aligned} & R(X,Y)W_8(Z,U)V = g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0 \\ & \Rightarrow g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y = 0 \\ & \Rightarrow W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y = 0 \\ & \Rightarrow g\big(W_8(Y,Z,U,V)X,\xi\big) - g\big(W_8(X,Z,U,V)Y,\xi\big) = 0 \\ & \Rightarrow W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) = 0 \end{aligned} \tag{3.3}$$

Note, this is only possible if $W_8'(Y,Z,U,V) = 0$ and $W_8'(X,Z,U,V) = 0$ since $A(X) \neq 0$ and $A(Y) \neq 0$ and thus follows the theorem.

Corollary 3.3: A $W_8 -$ semi-symmetric K-contact manifold is neither Einstein nor η -Einstein manifold.

4. $W_8 -$ symmetric K-Contact Riemannian manifold

Chaki and Gupta [6] gave the definition of a conformally symmetric manifold as for which $\nabla_U C = 0$ which is said to be conformally symmetric (where C is conformal curvature tensor).

Definition 4.1: A K-contact Riemannian manifold M is said to be $W_8 -$ symmetric if

$$\nabla_U W_8(X,Y)Z = 0 \tag{4.1}$$

Theorem 3.2: $W_8 -$ symmetric and a $W_8 -$ flat K-contact Riemannian manifold is a flat-manifold..

Proof: If the K-contact space is a $W_8 -$ symmetric and $W_8 -$ semi-symmetric then it follows

$$\begin{aligned} 0 = R(X,Y)W_8(Z,U)V) - W_8(R(X,Y)Z,U)V - W_8(Z,R(X,Y)U)V \\ - W_8(Z,U)R(X,Y)V \end{aligned} \tag{3.2}$$

Computing each of the above four terms separately yields

$$\begin{aligned} R(X,Y)W_8(Z,U)V) &= g(Y,W_8(Z,U)V)X - g(X,W_8(Z,U)V)Y \\ &= W_8'(Y,Z,U,V)X - W_8'(X,Z,U,V)Y \\ g\big(R(X,Y,W_8(Z,U)V,\xi\big) &= g\big(W_8'(Y,Z,U,V)X,\xi) - W_8'(X,Z,U,V)Y,\xi)\big) \\ &= \eta\big(W_8'(Y,Z,U,V)X\big) - \eta\big(W_8'(X,Z,U,V)Y\big) \\ &= W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) \end{aligned} \tag{3.3}$$

Again,

$$W_8(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1}\left[S(R(X,Y)Z,U)V - S(U,V)R(X,Y)Z\right] \quad (3.4)$$

$$= R(R(X,Y)Z,U)V + \frac{n-1}{n-1}\left[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\right]$$

$$= R(R(X,Y)Z,U)V + \left[g(R(X,Y)Z,U)V - g(U,V)R(X,Y)Z\right]$$

$$\begin{aligned} &= g(U,V)R(X,Y)Z - g(R(X,Y)Z,V)U + \left[R'(X,Y,Z,U)V - g(U,V)R(X,Y)Z\right] \\ &= R'(X,Y,Z,U)V - g(R(X,Y)Z,V)U \\ &= R'(X,Y,Z,U)V - R'(X,Y,Z,V)U \end{aligned}$$

$$\begin{aligned} W_8'(R(X,Y)Z,U,V,\xi) &= g\left(R'(X,Y,Z,U)V,\xi\right) - g\left(R'(X,Y,Z,V)U,\xi\right) \\ &= R'(X,Y,Z,U)\eta(V) - R'(X,Y,Z,V)\eta(U) \end{aligned}$$

Also,

$$\begin{aligned} W_8(Z,R(X,Y)U)V &= R(Z,R(X,Y)U)V + \frac{1}{n-1}\left[S(Z,R(X,Y)U)V - S(R(X,Y)U,V)Z\right] \\ &= g(R(X,Y)U,V)Z - g(Z,V)R(X,Y)U + \left[g(Z,R(X,Y)U)V - g(R(X,Y)U,V)Z\right] \\ &= -g(Z,V)R(X,Y)U + g(Z,R(X,Y)U)V \\ &= -g(Z,V)R(X,Y)U + R'(X,Y,U,Z)V \\ g\left(W_8'(Z,R(X,Y)U)V,\xi\right) &= g\left(R'(Z,X,Y,U)V,\xi\right) - g\left(g(Z,V)R(X,Y)U,\xi\right) \\ &= R'(X,Y,U,Z)\eta(V) - g(Z,V)R'(X,Y,U,\xi) \end{aligned} \quad (3.5)$$

$$\begin{aligned} W_8(Z,U)R(X,Y)V &= R(Z,U)R(X,Y)V + \frac{1}{n-1}\left[S(Z,U)R(X,Y)V - S(U,R(X,Y)V)Z\right] \\ &= g(U,R(X,Y)V)Z - g(Z,R(X,Y)V)U + \left[g(Z,U)R(X,Y)V - g(U,R(X,Y)V)Z\right] \\ &= g(Z,U)R(X,Y)V - g(Z,R(X,Y)V)U \\ &= g(Z,U)R(X,Y)V - R'(X,Y,V,Z)U \\ g\left(W_8(Z,U)R(X,Y)V,\xi\right) &= g\left(g(Z,U)R(X,Y)V,\xi\right) - g\left(R'(Z,X,Y,V)U,\xi\right) \\ &= g(Z,U)R'(X,Y,V,\xi) - R'(X,Y,V,Z)\eta(U) \end{aligned} \quad (3.6)$$

Next, we put together (3.3), (3.4), (3.5) and (3.6) to have

$$\begin{aligned} &W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) \\ &\quad -\{R'(X,Y,Z,U)\eta(V) - R'(X,Y,Z,V)\eta(U) \\ &\quad + R'(X,Y,U,Z)\eta(V) - g(Z,V)R'(X,Y,U,\xi) \\ &\quad + g(Z,U)R'(X,Y,V,\xi) - R'(X,Y,V,Z)\eta(U)\} = 0 \end{aligned} \quad (3.7)$$

Terms which are coefficients of $\eta(V)$ and $\eta(U)$ cancel out since R' is skew-symmetric with respect to the last two variables. Hence, (3.7) reduces to

$$\begin{aligned} &W_8'(Y,Z,U,V)\eta(X) - W_8'(X,Z,U,V)\eta(Y) + g(Z,V)R'(X,Y,U,\xi) \\ &- g(Z,U)R'(X,Y,V,\xi) = 0 \end{aligned} \quad (3.8)$$

but it is a $W_8 -$ flat manifold, hence $W_8{}' = 0$

Therefore (3.8) reduces to

$$(Z,V)R'(X,Y,U,\xi) - g(Z,U)R'(X,Y,V,\xi) = 0 \quad (3.9)$$

But in (3.9)

$$g(Z,U) \neq g(Z,V) \neq 0 \Rightarrow R' = 0 \quad (3.10)$$

Thus, follows the theorem.

5. A W_8 − Recurrent K-contact Riemannian manifold.

Definition 5.1: A K-contact Riemannian manifold is said to be recurrent if

$$(\nabla_U W_8)(X,Y)Z = B(U)W_8(X,Y)Z \quad (5.1)$$

Where B is a non-zero 1-form.

Theorem: 5.2: A W_8 -recurrent and W_8 -flat manifold is a flat manifold.

Proof; We have

$$(\nabla_U W_8)(X,Y)Z = B(U)W_8(X,Y)Z \quad \text{where } B(U) \neq 0 \quad (5.2)$$

but, if $W_8(X,Y)Z = 0$

Hence, (5.2) by definition becomes

$$0 = R'(X,Y,Z,U) + \frac{1}{n-1}\left[S(X,Y)g(Z,U) - S(Y,Z)g(X,U)\right] \quad (5.3)$$

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Y)g(Z,U)\right] \quad (5.4)$$

But, for a K-contact manifold

$$R'(X,Y,Z,U) = \left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right] \quad (5.5)$$

So (5.4) can only be true if and only if

$$R'(X,Y,Z,U) = 0$$

And therefore, the theorem follows.

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