International Journal of Statistics and Applied Mathematics 2017; 2(1): 64-65

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2017; 2(1): 64-65
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www.mathsjournal.com
Received: 20-11-2016
Accepted: 21-12-2016

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Conjugacy class sizes of some real elements and solvability of finite groups

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Abstract
Let G be a finite group. We mainly investigate how certain arithmetical conditions on conjugacy class sizes of all real elements of prime power orders of G influence the solvability of G. Some known results are generalized.
Mathematics Subject Classification: 20D15, 20E45

Keywords: Finite group, conjugacy class sizes, solvable groups

1. Introduction

Throughout the following, G always denotes a finite group. For an element x of G, we denote by x^G the conjugacy class of G in which x lies, $|x^G|$ the size of x^G (following Baer [1], we call $Ind_G(x)=|x^G|=|G:C_{G}(x)|$, the index of x in G). The rest of our notation and terminology are standard. The reader may refer to ref [6].

It is well known that there is a strong relation between the structure of a group and the sizes of its conjugacy classes and there exist several results studying the solvability of a group under some arithmetical conditions on its conjugacy class sizes (for example, see) [1-2, 4-5]. Among these results, a classical result of Chillag and Herzog [2] show that G is solvable if 4 does not divide any conjugacy class sizes of a finite group G. On the other hand, some other authors replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes to investigate the solvability of a finite group. For instance, Baer in [1] proves that a group G is solvable if its elements of prime power order have also prime power index. In [5], Shirong Li proves that G is solvable if 4 does not divide any conjugacy class sizes of elements of prime power order of a finite group G.

Recall that a conjugacy class g^G of G is called real if $g^G=(g^{-1})^G$, and the terminology is due to the following fact: these are precisely the conjugacy classes where every character of G takes a real value. It has been shown that even this relatively small set has a remarkable connection with the structure of the group. For instance, in [4] the authors prove that every group whose real classes all have prime power size is solvable the corollary of Theory 3.1 in [4]. Stimulated by the above results, in this paper, we will continue to investigate the influences of conjugacy class sizes of finite groups on the solvability of finite groups. We show how imposing some arithmetical conditions on conjugacy class sizes of some real elements of prime power orders of G yields restrictions on the solvability of G. Our main result is the following:

Theorem A. If all real elements of prime power order in G have also prime power index, then G is solvable.

2. Preliminaries

In this section, we give some lemmas which are useful for our main results.

Lemma 2.1 ([7, Lemma 6]) Let N be a normal subgroup of a group G and let p be an odd prime. If Nx is a p-power order real element in G/N, then Nx=Ny for some p-power order real element y in G.

Lemma 2.2 ([4, Lemma 2.4]) Let G be a group.

(i) If x is a real element of G and $|x^G|$ is odd, then $x^2=1$.
(ii) If x is a real element of G, then every power of x is a real element of G.
(iii) The identity is the unique real element of G if and only if |G| is odd.
(iv) If $N \lhd G$ and |G/N| is odd, then Re (G)=Re(N).

Lemma 2.3 ([2, Theorem 1.1]) Let $N \lhd G$, $x \in N$ and $y \in G$.Then
(i)$|x^N| \mid |x^G|$.
(ii)$|yN^{G/N}| \mid |y^G|$.

Lemma 2.4 The group G has no nontrivial real elements of odd prime power order if and only if G has a normal Sylow 2-subgroup.

Proof. By the proof of Proposition 6.4 in [3], we can easily prove the Lemma.

3. Proof of the Main Theorem

Proof of Theorem A: We argue by induction on |G|. Let N be a maximal normal subgroup of G. Recalling that every real element x of prime power order of N is a real element of G, and that $|x^N|$ divides $|x^G|$ by Lemma 2.3, we get that, for every $x \in$ Re (N), $|x^N|$ is a prime power. By induction, N is solvable.
Now we assume that G/N is a nonabelian simple group. By Feit-Thompson's Odd Order Theorem and Lemma 2.4, there exists a nontrivial $xN \in$ Re (G/N) of odd prime power order. Then, by (i) of Lemma 2.2, $|xN^{G/N}|$ is even. Now, by Lemma 2.1, there exists $y \in$ Re (G) of prime power order such that yN=xN and, since $|xN^{G/N}|=|yN^{G/N}|$ divides $|y^G|$, it follows that $|xN^{G/N}|$ is a prime power. This contradicts a classical result by Burnside, and we conclude that G/N must be solvable. The desired conclusion follows. Our proof is complete now.

4. Acknowledgements

The research is supported by the NNSF of China (11301138).The paper is dedicated to Professor John Cossy for his 75th birthday.

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