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Analysis of a discrete model of prey-predator system with prey refuge

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Abstract
A refuge model of prey-predator system with Holling type-II functional response is proposed. The existence of fixed points is established and their stability conditions are derived. Neimark-Sacker bifurcation result is obtained. Numerical simulations suggest that the system exhibits chaotic behavior under certain situations along with Fold and Flip bifurcation.

Keywords: prey-predator system, refuge, neimark-sacker bifurcation, white noise

Introduction

The dynamical behavior of prey-predator interaction creates major interest over a long period of time. Lot of works has been done on prey-predator system with different types of functional responses. It is observed that predation pressure sometimes influences prey population to take refuge. In natural ecosystem, prey may avoid killed by their predators and as a result they defend themselves by making refuge in different ways. The refuge habitat consists of burrows [1], trees [2], cliff faces [3], thick vegetation [4] or rock talus [5] etc. and in aquatic ecosystem benthic coral cover provides the refuge for prey fish in pristine coral reefs [6]. A wild life refuge may be a naturally occurring sanctuary, such as an island, which provides protection for species from hunting, predation or competition, or it may refer to a protected area, a geographic territory within which wildlife is protected. Particularly in mite prey-predator interactions, there often exist spatial refugia that can afford the prey some degree of protection from predation and therefore reduce the chance of extinction due to predation. Some of the empirical works have investigated the effect of prey refuge. In 1934, Gause [7] found that in the experiments with Didinium Nausatum (predator) and Paramecium caudatum (prey), D. Nausatum overexploited P. Caudatum leading first to its extinction and subsequently to its own. However, his further work showed that a prey-predator community could be self-sustaining if there were refuges for the prey population.

Effect of refuges on stability of prey-predator interactions have been analyzed in [8, 9]. Sarwardi *et al.* [10] considered a model incorporating a constant proportion of prey refuge consisting of two prey and one predator population with the inclusion of Holling type-II response function and analyzed local and global stability of the system together with Hopf bifurcating periodic solutions. Das *et al.* [11] studied a prey-predator model with Holling type II functional response incorporating constant prey refuge and harvesting to both prey and predator species and discussed the local as well as global stabilities at interior equilibrium of the system. Occurrence of Hopf bifurcation of the system is also analyzed. Authors mentioned the impact of prey refuge and harvesting efforts and also suggested for stochastic analysis from realistic point of view. Chakraborty *et al.* [12] studied a prey-predator system obeying logistic law of growth with constant prey refuge through provision of alternative food to predators. Authors analyzed the variability of the system in presence of constant prey refuge and examined the stabilizing effect on prey-predator system.

They further remarked that the effects of refuges can stabilize the model system and destabilize it under certain conditions.

The above articles show that prey-predator model with prey refuge are mainly studied in continuous system. Actually for non-overlapping generations, the prey-predator system can be modeled in a discrete from. Discrete time models help in numerical implication. It also provides rich dynamics than a continuous-time model of the same time. The discrete Leslie-Gower predator-prey model with constant population of prey refuge is studied by Zhuang and Wen [14]. But none of the studies in discrete prey-predator model with a constant number of prey using refuges is addressed. Further ecological fluctuations and other factors in stochastic environment are also important for this prey-predator system with refuge. The study of consequences of the refuge is more a topic of interest in theoretical ecology. Although much work has been done, still many problems are unsolved. Hence mathematical analysis of such systems requires the efforts of both mathematicians and ecologists. As refuge of prey is a natural phenomenon, we consider it in our model. The main thrust of this paper is to construct the discrete predator-prey model with Holling type-II response function incorporating a constant prey refuge and to study its dynamics.

Mathematical Model

Gonzalez-Olivares and Ramos-Jiliberto [15] studied the dynamical consequences of the following predator-prey systems with constant number of prey using refuges which protects m number of prey from predation:

$$\frac{dx}{dt} = r(1-\frac{x}{k})x - bf(x)y \tag{1}$$

$$\frac{dy}{dt} = -dy + cf(x)y, \ \ where$$

$$f(x) = 0, x \le m$$

$$= \tfrac{x-m}{a+x-m}, x > m$$

where x and y denote prey and predator population sizes, respectively, at any time $t. r > 0$ represents the intrinsic growth rate of prey, k is the carrying capacity of the prey in the absence of predator, m is the constant number of prey using refuges, which protect m of prey from predation, $b > 0$ is the per capita predator consumption rate, $c > 0$ is the efficiency with which predators convert consumed prey into new predator, $a > 0$ is the half saturation constant and $d > 0$ is the death rate of the predator.

When $x \le m$, predator dies exponentially i.e. the model (1) becomes

$$\frac{dx}{dt} = r(1-\frac{x}{k})x \tag{2}$$

$$\frac{dy}{dt} = -dy$$

Further when $x > m$, the population dynamics of (1) are given by the following:

$$\frac{dx}{dt} = r(1-\frac{x}{k})x - \frac{b(x-m)y}{a+x-m} \tag{3}$$

$$\frac{dy}{dt} = -dy + \frac{c(x-m)y}{a+x-m}$$

For system (1) local stability property and the existence of the limit cycle are studied. The influences of the refuge are also discussed. We now present the discrete-time predator-prey system with constant number of prey using refuges as follows:

$$x_{n+1} = x_n + r(1-\frac{x_n}{k})x_n - \frac{b(x_n-m)y_n}{a+x_n-m} \tag{4}$$

$$y_{n+1} = y_n - dy_n + \frac{c(x_n-m)y_n}{a+x_n-m}$$

Linear analysis & results

In this section we first determine the existence of fixed points of system (4) and then investigate their stability by calculating the eigenvalues for the variational matrix of system (4) at each fixed points. From system (4) we obtained three fixed points namely $E_0(0,0), E_1(k,0)$ and $E_2(x^*, y^*)$ where $x^* = \frac{m(c-d)+ad}{c-d}$, $y^* = \frac{crx^*}{bd}(1-\frac{x^*}{k})$. E_2 is feasible if $c > d$ and $k > x^*$.

Now the Jacobian matrix of system (4) at a fixed point (x, y) is as follows:

$$J(x,y) = \begin{pmatrix} 1+r(1-\frac{2x}{k})-\frac{aby}{(a+x-m)^2} & -\frac{b(x-m)}{a+x-m} \\ \frac{acy}{(a+x-m)^2} & 1-d+\frac{c(x-m)}{a+x-m} \end{pmatrix}$$

1. At the fixed point $E_0(0,0)$ the eigenvalues of the Jacobian matrix of system (4) are $\lambda_1 = 1+r$ and $\lambda_2 = 1-d-\frac{cm}{a-m}$. Since $\lambda_1 > 1$, the fixed point E_0 is unstable and E_0 is saddle if $d+\frac{cm}{a-m} < 1$.
2. At the fixed point $E_1(k,0)$ the eigenvalues of the Jacobian matrix of system (4) are $\lambda_1 = 1-r$ and $\lambda_2 = 1-d+\frac{c(k-m)}{a+k-m}$. Hence E_1 is stable when $r < 1$ and $\frac{c(k-m)}{a+k-m} < d$.
3. At the fixed point $E_2(x^*, y^*)$

$$J(E_2) = \begin{pmatrix} 1+r(1-\frac{2x^*}{k})-\frac{aby^*}{(a+x^*-m)^2} & -\frac{bd}{c} \\ \frac{acy^*}{(a+x^*-m)^2} & 1 \end{pmatrix}$$

Now $trJ(E_2) = 2+r(1-\frac{2x^*}{k})-\frac{aby^*}{(a+x^*-m)^2}$ and $\det J(E_2) = 1+r(1-\frac{2x^*}{k})+\frac{aby^*(d-1)}{(a+x^*-m)^2}$.

Hence Neimark-Sacker bifurcation occurs when $\det J(E_2) = 1$ and $-2 < trJ(E_2) < 2$. Further Saddle-node(fold) bifurcation and Flip bifurcation occur respectively

for $\det(J(E_2))-tr(J(E_2))+1>0$ and $\det(J(E_2))+tr(J(E_2))+1>0$.

Thus we have the following Results in Theorem 1:

Theorem 1. (i) E_0 is saddle if $d+\frac{cm}{a-m}<1$.

(ii) E_1 is stable when $r<1$ and $\frac{c(k-m)}{a+k-m}<d.$

(iii) If $r(1-\frac{2x^*}{k})+\frac{aby^*(d-1)}{(a+x^*-m)^2}=0$ and $abdy^*<4(a+x^*-m)^2$ hold simultaneously a Neimark-Sacker bifurcation occurs near E_2.

(iv) Again saddle-node (fold) bifurcation occurs near the equilibrium point for $\det(J(E_2))-tr(J(E_2))+1>0$ and period-doubling (Flip) bifurcation arises if $\det(J(E_2))+tr(J(E_2))+1>0$.

Conclusion with numerical simulations

We have proposed and analyzed the discrete version of the prey-predator model incorporating constant prey refuge. We have obtained a Neimark-Sacker bifurcation in this system in place of Hopf bifurcation which is generally obtained in continuous system and several other rich dynamics are observed including chaos.

From numerical simulations it is observed that for a choice of hypothetical set of parametric values the continuous system is stable around the point (1,1) (see Fig. 1) whereas the Neimark-Sacker bifurcation occurred near the equilibrium point (1,1) in the discrete system (see Fig. 2). Fig. 3 and Fig. 4 vividly depict that as number of prey using refuges (m) increases, the system switches from instability to stability. In Fig. 5, m is a bifurcation parameter varies from 2 to 2.5 and exhibits fold bifurcation. Hence the parameter m has an important role to stabilize the system. Further from Fig. 6 and Fig. 7, we observed a period-halving bifurcation leading to order followed by period doubling bifurcation leading to chaos where r plays a vital role in this chaotic behavior of the system. In figure 8, Flip bifurcation is observed and further the chaotic behavior is illustrated in Figure 9.

Further the effect of environmental noise on the discrete model is observed in system (4) with respect to white noise perturbation around its positive Equilibrium by maintaining the tolerance limit of its intensity. From numerical simulation it is numerical simulation it is observed that the system is chaotic (see Fig. 10) with intensity of the white noise equals to 0.2.

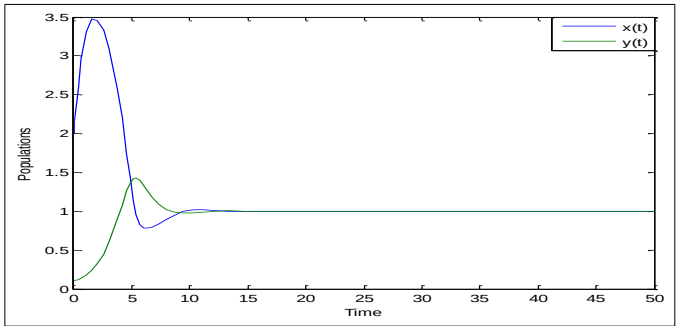

Fig 1: The figure illustrates the trajectories of the system for r = 2; k = 4; b = 4.5; m = 0.5; a = 1; c = 1.5; d =0.5 with initial point (2, 0.1).

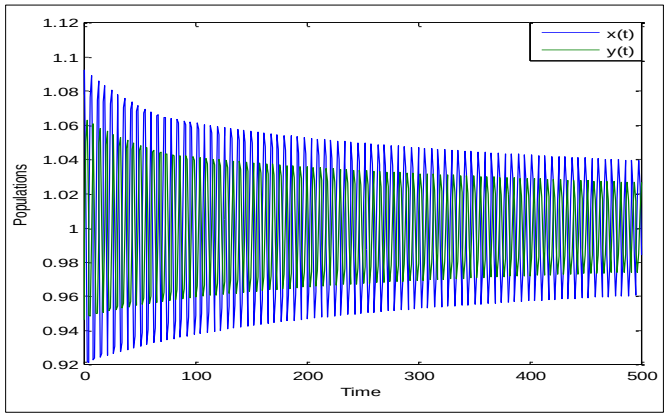

Fig 2: The trajectories illustrate the Neimark-Shaker bifurcation in the system with r = 2; k = 4; b = 4.5; m = 0.5; a = 1; c = 1.5; d =0.5.

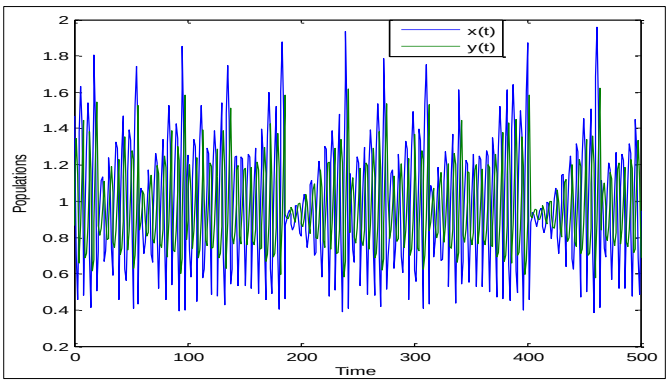

Fig 3: The figure illustrates the trajectories of the system for r = 2; k = 4, b = 4.5; m = 0.4; a = 1; c = 1.5; d =0.5.

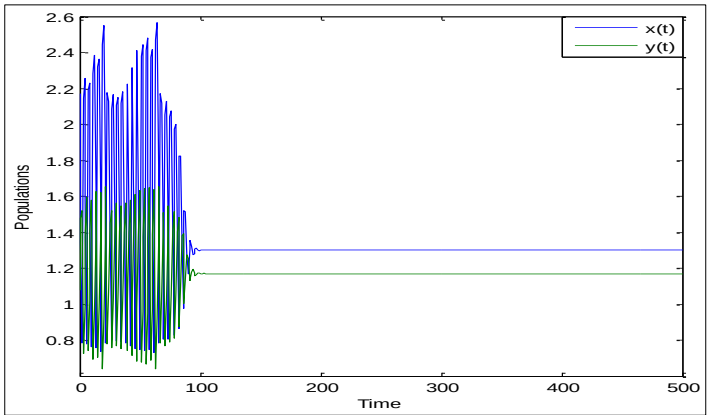

Fig 4: The figure illustrates the trajectories of the system for r = 2; k = 4; b = 4.5; m = 0.8; a = 1; c = 1.5; d =0.5.

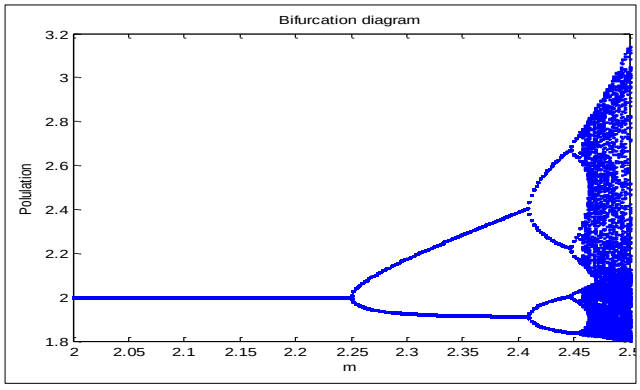

Fig 5: The figure illustrates the fold bifurcations occurs in the system for m $\in$ [2, 2.5] with other parameter values r = 2; k = 4; b = 4.5; a = 1; c = 1.5; d = 0.5 and initial point (2, 0.1).

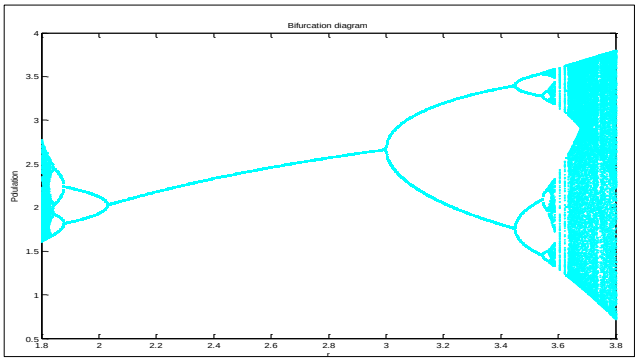

Fig 6: The diagram shows the fold bifurcation occurs in the system for r ∈ [1.8, 3.8] with other parameter values k = 4; b = 4.5; a = 1; m = 2.28; c = 1.5; d = 0.5 and initial point (2, 0.01).

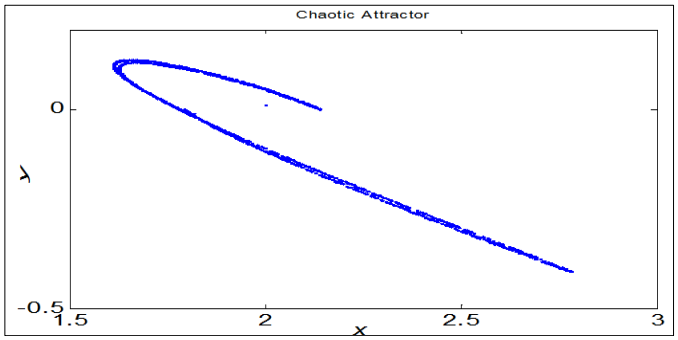

Fig 7: The diagram shows the chaotic attractor appears in the system for r ∈ [1.8, 3.8] with other parameter values k = 4; b = 4.5; a = 1; m = 2.28; c = 1.5; d = 0.5.

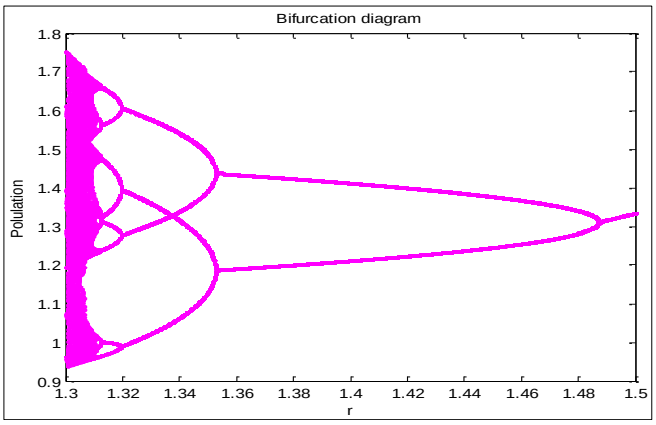

Fig 8: The figure illustrates flip bifurcation in the system for r ∈ [1.3, 1.5] with other parameter values k = 4; b = 4.5; m = 1.5; a = 1; c = 1.5; d = .65 and initial point (2, 1).

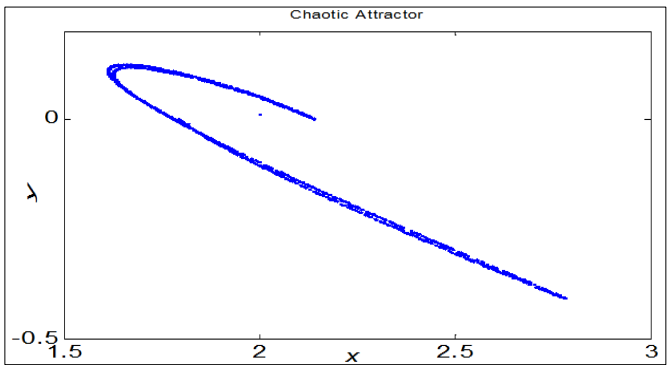

Fig 9: The diagram shows the chaotic attractor appears in the system for r ∈ [1.3, 1.5] with other parameter values k = 4; b = 4.5; m = 1.5; a = 1; c = 1.5; d = .65.

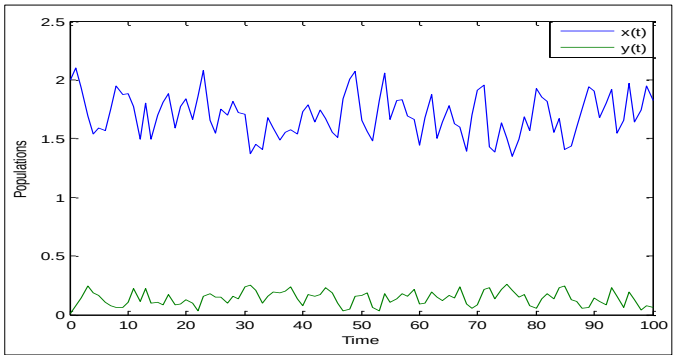

Fig 10: The figure illustrates the chaos in the system for r = 2; k = 4; b = 4.5; m = 0.5; a = 1; c = 1.5; d =0.5 with initial point (2, 0.01). The intensity of the noise is 0.2.

In brief, in this article, we have identified two bifurcation parameter intrinsic growth rate (r) and number of prey using refuges (m). Both r and m play important role over the stable as well as chaotic behavior of the system. We observe that using refuges prey increases their protection from predation and that yields the system becomes stable. Further effect of environmental noise on the discrete model is also remarkable.

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