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# Explicit expression for a first integral and phase portrait for a class of planar differential system

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**Abstract**
In this paper we study the integrability of two-dimensional autonomous systems in the plane of the form

$$\begin{cases} x' = P(x,y)\sin\left(\dfrac{A_1(x,y)}{A_2(x,y)}\right) + x\cos\left(\dfrac{R(x,y)}{S(x,y)}\right), \\ y' = Q(x,y)\sin\left(\dfrac{B_1(x,y)}{B_2(x,y)}\right) + y\cos\left(\dfrac{R(x,y)}{S(x,y)}\right), \end{cases}$$

where $A_1(x,y)$, $A_2(x,y)$, $B_1(x,y)$, $B_2(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree a, a, b, b, n, n, m, m respectively. Writing this system in polar coordinates, from which we can compute a formal First integral for the system. Finally, we give the curves which are formed by the trajectories for the system.

**Keywords:** Autonomous Systems, First Integral, Curves, Phase portrait
**2010 Mathematics Subject Classification**. 34C07, 37K10.

## 1. Introduction

We consider two-dimensional autonomous systems of differential equations of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = F(x,y), \\ y' = \dfrac{dy}{dt} = G(x,y). \end{cases} \quad \text{..........................(1)}$$

where $F(x,y)$ and $G(x,y)$ are reals functions. In the qualitative theory of planar dynamical systems [1, 8, 9], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [18]. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly [2, 3, 6, 11, 13]. There exist three main open problems in the qualitative theory of real planar differential systems, the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals. Giné and Llibre characterized a large classes of polynomial differential systems in terms of the existence of first integrals [4, 7, 9, 12, 16, 17, 21]. For more details about first integral see for instance [10, 14, 15, 19, 20], see the references quoted in those articles. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1).

System (1) is integrable on an open set $\Omega$ of $\mathbb{R}^2$ if there exists a non-constant $C^1$ function $H: \Omega \to \mathbb{R}$, called a first integral of the system on $\Omega$, which is constant on the trajectories of the system (1) contained in $\Omega$ i.e. if

$$\frac{dH(x,y)}{dt}=\frac{\partial H(x,y)}{\partial x}F(x,y)+\frac{\partial H(x,y)}{\partial y}G(x,y)\equiv 0,$$

in the points of $\Omega$.
Moreover, H=h is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait see [5].
In this paper we study the integrability of two-dimensional autonomous systems in the plane of the form

$$\begin{cases} x'=P(x,y)\sin\left(\frac{A_1(x,y)}{A_2(x,y)}\right)+x\cos\left(\frac{R(x,y)}{S(x,y)}\right), \\ y'=Q(x,y)\sin\left(\frac{B_1(x,y)}{B_2(x,y)}\right)+y\cos\left(\frac{R(x,y)}{S(x,y)}\right), \end{cases} \quad ..................(2)$$

where $A_1(x,y)$, $A_2(x,y)$, $B_1(x,y)$, $B_2(x,y)$, $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$ are homogeneous polynomials of degree a, a, b, b, n, n, m, m respectively. Writing this system in polar coordinates, from which we can compute a formal First integral for the system. Finally, we give the curves which are formed by the trajectories for the system.
We define the trigonometric functions

$$f_1(\theta)=P(\cos\theta,\sin\theta)(\cos\theta)\sin\frac{A_1(\cos\theta,\sin\theta)}{A_2(\cos\theta,\sin\theta)}+Q(\cos\theta,\sin\theta)(\sin\theta)\sin\frac{B_1(\cos\theta,\sin\theta)}{B_2(\cos\theta,\sin\theta)},$$

$$f_2(\theta)=\cos\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)},$$

$$f_3(\theta)=Q(\cos\theta,\sin\theta)(\cos\theta)\sin\frac{B_1(\cos\theta,\sin\theta)}{B_2(\cos\theta,\sin\theta)}-P(\cos\theta,\sin\theta)(\sin\theta)\sin\frac{A_1(\cos\theta,\sin\theta)}{A_2(\cos\theta,\sin\theta)}.$$

**Main result**
Our main result on the existence of a First integral and the curves which are formed by the trajectories of the 2-dimensional differential systems (2) is the following.

**Theorem 1** *Let us consider the system* (2)*, then the following statements hold.*

(a) *If $f_3(\theta)\neq 0$, $n\neq 1$ and $S(\cos\theta,\sin\theta)\,A_2(\cos\theta,\sin\theta)\,B_2(\cos\theta,\sin\theta)\neq 0$, then system* (2) *has the first integral*

$$H(x,y)=\left(x^2+y^2\right)^{\frac{n-1}{2}}\exp((1-n)\int_0^{\arctan\frac{y}{x}}g_1(s)ds)-$$

$$(n-1)\int_0^{\arctan\frac{y}{x}}\exp((1-n)\int_0^{w}g_1(s)ds)g_2(w)dw,$$

*where* $g_1(\theta)=\frac{f_1(\theta)}{f_3(\theta)}, g_2(\theta)=\frac{f_2(\theta)}{f_3(\theta)}.$

*Moreover the phase portrait of the differential system* (2)*, in Cartesian coordinates is given by*

$$x^2+y^2=\begin{pmatrix} h\exp((n-1)\int_0^{\arctan\frac{y}{x}}g_1(s)ds)+ \\ (n-1)\exp((n-1)\int_0^{\arctan\frac{y}{x}}g_1(s)ds) \\ \int_0^{\arctan\frac{y}{x}}\exp((1-n)\int_0^{w}g_1(s)ds)g_2(w)dw \end{pmatrix}^{\frac{2}{n-1}},$$

*where $h\in\mathbb{R}$.*

(b) *If $f_3(\theta)\neq 0$, $n=1$ and $S(\cos\theta,\sin\theta)\,A_2(\cos\theta,\sin\theta)\,B_2(\cos\theta,\sin\theta)\neq 0$, then system* (2) *has the first integral*

$$H(x,y)=\left(x^2+y^2\right)\exp(-\int_0^{\arctan\frac{y}{x}}(g_1(s)+g_2(s))ds).$$

Moreover the phase portrait of the differential system (2), in Cartesian coordinates is given by

$$x^2+y^2=h\exp(\int_0^{\arctan\frac{y}{x}}(g_1(s)+g_2(s))ds),$$

*where $h\in\mathbb{R}$.*

(c) *If $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$, then system* (2) *has the first integral* $H(x,y)=\frac{y}{x}.$

*Moreover the phase portrait of the differential system* (2)*, in Cartesian coordinates is given by y=hx where $h\in\mathbb{R}$.*

**Proof** In order to prove our results we write the polynomial differential system (2) in polar coordinates (r,θ) defined by x=rcosθ and y=rsinθ, then system (2) becomes

$$\begin{cases} r'=f_1(\theta)r^n+f_2(\theta)r, \\ \theta'=f_3(\theta)r^{n-1}, \end{cases} \quad (3)$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r'=\frac{dr}{dt},\theta'=\frac{d\theta}{dt}.$

*If $f_3(\theta)\neq 0$, $n\neq 1$ and $S(\cos\theta,\sin\theta)\,A_2(\cos\theta,\sin\theta)\,B_2(\cos\theta,\sin\theta)\neq 0$.*

Taking as new independent variable the coordinate θ, this differential system (3) writes

$$\frac{dr}{d\theta}=g_1(\theta)r+g_2(\theta)r^{2-n} \quad (4)$$

Where $g_1(\theta)=\frac{f_1(\theta)}{f_3(\theta)}, g_2(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho=r^{n-1}$ we obtain the linear equation

$$\frac{d\rho}{d\theta}=(n-1)\left(g_1(\theta)\rho+g_2(\theta)\right) \quad (5)$$

The general solution of linear equation (5) is

$$\rho(\theta)=\exp((n-1)\int_0^{\theta}g_1(s)ds)\left[\alpha+(n-1)\int_0^{\theta}\exp((1-n)\int_0^{w}g_1(s)ds)g_2(w)dw\right],$$

where $\alpha\in\mathbb{R}$., which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n-1}{2}} \exp((1-n)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) - (n-1)\int_0^{\arctan\frac{y}{x}} \exp((1-n)\int_0^{w} g_1(s)ds) g_2(w)dw.$$

The curves H=h with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), are written in Cartesian coordinates as

$$x^2+y^2 = \left( \begin{array}{l} h\exp((n-1)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) + \\ (n-1)\exp((n-1)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) \\ \int_0^{\arctan\frac{y}{x}} \exp((1-n)\int_0^{w} g_1(s)ds) g_2(w)dw \end{array} \right)^{\frac{2}{n-1}},$$

where $h\in\mathbb{R}$.

Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $n=1$ and $S(\cos\theta,\sin\theta) A_2(\cos\theta,\sin\theta) B_2(\cos\theta,\sin\theta)\neq 0$.

Taking as new independent variable the coordinate $\theta$, this differential system (3) writes

$$\frac{dr}{d\theta} = \left(g_1(\theta)+g_2(\theta)\right)r \tag{6}$$

where $g_1(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}, g_2(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}.$

The general solution of equation (6) is

$$\rho(\theta) = \exp((n-1)\int_0^{\theta} g_1(s)ds)\left[\alpha + (n-1)\int_0^{\theta} \exp((1-n)\int_0^{w} g_1(s)ds) g_2(w)dw\right],$$

where $\alpha\in\mathbb{R}$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n-1}{2}} \exp((1-n)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) - (n-1)\int_0^{\arctan\frac{y}{x}} \exp((1-n)\int_0^{w} g_1(s)ds) g_2(w)dw.$$

The curves H=h with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), are written in Cartesian coordinates as

$$x^2+y^2 = \left( \begin{array}{l} h\exp((n-1)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) + \\ (n-1)\exp((n-1)\int_0^{\arctan\frac{y}{x}} g_1(s)ds) \\ \int_0^{\arctan\frac{y}{x}} \exp((1-n)\int_0^{w} g_1(s)ds) g_2(w)dw \end{array} \right)^{\frac{2}{n-1}},$$

where $h\in\mathbb{R}$.

Hence statement (b) of Theorem 1 is proved.

Assume now that $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$, then from (3) it follows that $\theta'=0$ So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system.

The curves H=h with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (2), are written in Cartesian coordinates as y=hx where $h\in\mathbb{R}$.

This completes the proof of statement (c) of Theorem 1. ■

**Conclusion**

The elementary method used in this paper seems to be fruitful to investigate planar differential systems of ODEs in order to obtain explicit expression for a first integral and characterizes its trajectories; this is a one of the classical tools in the classification of all trajectories of dynamical systems.

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