International Journal of Statistics and Applied Mathematics 2017; 2(2): 41-42

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2017; 2(2): 41-42

www.mathsjournal.com
Received: 21-01-2017
Accepted: 22-02-2017

Qiang Zhou
College of Mathematics and Information Science,
Xinyang Normal University,
Xinyang, People's Republic of China

Hong Pan
College of Mathematics and Information Science,
Xinyang Normal University,
Xinyang, People's Republic of China

Correspondence
Qiang Zhou
College of Mathematics and Information Science,
Xinyang Normal University,
Xinyang, People's Republic of China

On semi cover-avoiding subgroups of finite groups

Qiang Zhou and Hong Pan

Abstract
In this paper, we obtain the supersolvability for a finite group based on the assumption that minimal subgroups and cyclic subgroups of order 4 have the semi cover-avoiding properties. Some known results are generalized.

Keywords: Finite Group, Semi cover-avoiding subgroups, Supersolvable groups.
Mathematics Subject Classification: 20D15, 20E45

1. Introduction

All groups considered in this paper are finite groups. Our notation and terminology are standard. The reader may refer to ref [5]. In 1962, Gaschutz [1] introduced a certain conjugacy class of subgroups of a finite solvable group which he called pre-Frattini subgroups. These subgroups have the property that they not only avoid the complemented chief factors of a finite solvable group G but also cover the rest of its chief factors, we call these subgroups that have cover-avoiding properties, that is, suppose that H$\leq$G, for any chief series 1=$G_0 < G_1 < \cdots < G_m$=G, such that for every i=1,$\cdots$,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}. Thereafter, many authors studies this property, for example, In 1993, Ezquerro [2] gave some characterization for a finite group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some Sylow subgroups of G have the cover-avoiding properties. Guo Xiuyun and K.P. Shum in [3] obtain some characterizations for a finite solvable group based on the assumption that some of its maximal subgroups or 2-maximal subgroups have the cover-avoiding properties.

Recently, in [4], Fan, Guo and Shum generalized the cover-avoiding properties of finite groups. They called Semi cover-avoiding properties, that is, a subgroup H is said to be semi cover-avoiding in a group G if there is a chief series 1=$G_0 < G_1 < \cdots < G_m$=G, such that for every i=1,$\cdots$,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}. They used Semi cover-avoiding properties of Sylow and maximal subgroups to investigate the solvability of finite groups.

In this paper, we will push further this approach and obtain the supersolvability for a finite group based on the assumption that minimal subgroups and cyclic subgroups of order 4 have the semi cover-avoiding properties. Our main result is the following:

Theorem A. Let G be a finite group. N$\lhd$G and G/N is supersolvable. If every minimal subgroup and every cyclic subgroup of order 4 of N are semi cover-avoiding subgroups of G, then G is supersolvable.

2. Basic definitions and preliminary results

In this section, we give one definition and some lemmas which are useful for our main results.

Definition 2.1 ([4]) A subgroup H is said to be semi cover-avoiding in a group G if there is a chief series 1=$G_0 < G_1 < \cdots < G_m$=G, such that for every i=1,$\cdots$,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}.

The following Lemma is obervous

Lemma 2.2 Let H be a subgroup of a group G and let $1<\cdots<N<\cdots<M<\cdots<G$ be a normal series of G. If the subgroup H covers (respectively avoids) M/N, then H covers (respectively avoids) any quotient factor between N and M of any refinement of the normal series.

Lemma 2.3 Let G be a group. Let H be a semi cover-avoiding subgroup of G and N be a normal subgroup of G.

1. If $H\leq K\leq G$, then H is a semi cover-avoiding subgroup of K.
2. If $N\subseteq H$ or gcd(|H|,|N|) = 1, where gcd(-, -) denotes the greatest common divisor, then HN/N is a semi cover-avoiding subgroup of G/N.

Proof. (i)Since H is a semi cover-avoiding subgroup of G, that is, there exists a chief series $1=G_0<G_1<\ldots<G_m=G$, such that for every i=1, $\ldots$, m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}. Suppose that $K_i=G_i\cap K$, i=1,$\ldots$,m. Then $HK_i=K_iH$ or $H\cap K_i=K_i\cap H$. So H semi covers or avoids a normal series $1=K_n\leq K_{n-1}\leq\ldots\leq K_0=K$. By Lemma 2.2 we know, H is a semi cover-avoiding subgroup of K.

(ii) By Lemma 2.2 in [4], we can prove the result.

Lemma 2.4 ([6]) Suppose that G is an inner-supersolvable group, then

1. there exists a normal $P\in Sly_p(G)$ such that $G=P\times M$, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.
2. if p>2, then expP=2; if p=2, then exp$P\leq 4$ and $p^2\,\|\,|G|$.
3. there exists c\in $P\backslash\Phi(P)$ such that $\langle c\rangle$ is not normal G.
4. if P is Abelian, then $\Phi(P)=1$.
5. if P is not Abelian, then $\Phi(P)=Z(P)=P'$.
6. G is a group with Sylow tower and G is an inner-nilpotent group.

3. Proof of the Main Theorem

Proof of Theorem A. Assume that the theorem is false and let G be a counterexample of minimal order.

- G is an inner-supersolvable group.

In fact, for any proper subgroup H of G, we have that $HN/N\cong H/H\cap N$. Since G/N is supersolvable, we know that $H/H\cap N$ is supersolvable. By Lemma 2.3 we know that every minimal subgroup and every cyclic subgroup of order 4 are semi cover-avoiding subgroups of H. So H, $H\cap N$ satisfy the hypotheses of the theorem. By the minimality of G, we know that H is supersolvable. That is, G is an inner-supersolvable group. By lemma 2.4 we can get that

1. there exists a normal $P\in Syl_p(G)$ such that $G=P\times M$, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.
2. (2)if p>2, then expP=2; if p=2, then exp$P\leq 4$ and $p^2\,\|\,|G|$.

- (ii)$P\in Syl_p(N)$.

In fact, since $G/P\cong M$ and M is supersolvable, we have that G/P is also supersolvable. By hypothesis of the theorem, we know that G/N is supersolvable. So $G/P\cap N$ is supersolvable. If $P\cap N\leq\Phi(P)$, then $G/\Phi(P)$ is supersolvable. So its quotient group $G/\Phi(G)$ is supersolvable, a contradiction. Thus $P\cap N$ can not be contained in $\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we have that $(P\cap N)\backslash\Phi(P)/\Phi(P)$ is a normal p-subgroup of $G/\Phi(P)$. So $P=(P\cap N)\Phi(P)=(P\cap N)\in Syl_p(N)$.

- (iii) G is supersolvable.

Suppose that an element $x\in P\backslash\Phi(P)$, then o(x)=p or o(x)=4. By hypothesis of the theorem, we know that $\langle x\rangle$ is a semi cover-avoiding subgroup of G. That is, there exists a chief series $1=G_0<G_1<\ldots<G_m=G$ such that $\langle x\rangle$ covers or avoids every G_{i+1}/G_i. Since $x\in G$, there exists an integer k such that x is not contained in G_k but $x\in G_{k+1}$. Since $G_k\cap\langle x\rangle\neq G_{k+1}\cap\langle x\rangle$, we have that $G_k\langle x\rangle=G_{k+1}\langle x\rangle=G_{k+1}$. So G_{k+1}/G_k is a cyclic group of order p or 4. Since $P\cap G_k$ is normal in G and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$, we have that $(P\cap G_k)\Phi(P)=\Phi(P)$ or P. If $(P\cap G_k)\Phi(P)=P$, then $P\cap G_k=P$, this is contrary to the fact x is not contained in $P\cap G_k$. So $P\cap G_k\leq\Phi(P)$. Consider normal subgroup If $(P\cap G_k)\Phi(P)$, we have that If $(P\cap G_k)\Phi(P)=P$. So $P/\Phi(P)$ is a cyclic group of order p or 4. By Lemma 2.3 and the minimality of G, we have that $G/\Phi(P)$ is supersolvable. So G is supersolvable, a contradiction. Our proof is complete now.

Acknowledgements

The research is supported by the NNSF of China(11201400).The paper is dedicated to Professor John Cossy for his 75th birthday.

References

1. Gaschutz W. Praefrattin gruppen. Arch. Math. 1962; 13:418-426.
2. Ezquerro LM. A contribution to the theory of finite supersolvable groups. Rend. Sem. Mat. Univ. Padova, 1993; 89:161-170.
3. Guo XY, Shum KP. Cover-avoidance property and the structure of finite groups. J. Pure Appl. Algebra. 2003; 181:297-308.
4. Fan Y, Guo XY, Shum KP. Remarks on two generalizations of normality of subgroups. Chinese Annals of Mathematics, 2006; 27A(2):169-176.
5. Robinson DJS. A Course in the Theory of Groups. Spring-verlag, New York-Heidelberg-Berlin. 1980.
6. Wang Pinchao, Wen Fengtong, Wexiang Li. C-normal subgroups of a finite group. Journal of Qufu Normal University, 1997; 23(4):5-7.