On semi cover-avoiding subgroups of finite groups

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Abstract
In this paper, we obtain the supersolvability for a finite group based on the assumption that minimal subgroups and cyclic subgroups of order 4 have the semi cover-avoiding properties. Some known results are generalized.

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1. Introduction
All groups considered in this paper are finite groups. Our notation and terminology are standard. The reader may refer to ref [5]. In 1962, Gaschutz [1] introduced a certain conjugacy class of subgroups of a finite solvable group which he called pre-Frattini subgroups. These subgroups have the property that they not only avoid the complemented chief factors of a finite solvable group G but also cover the rest of its chief factors, we call these subgroups that have cover-avoiding properties, that is, suppose that H≤G, for any chief series 1= 0

G

< 1

G

< \ldots < mG =G, such that for every i=1,\ldots ,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}.

Thereafter, many authors studies this property, for example, In 1993, Ezquerro [2] gave some characterization for a finite group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some Sylow subgroups of G have the cover-avoiding properties. Guo Xiuyun and K.P. Shum in [3] obtain some characterizations for a finite solvable group based on the assumption that some of its maximal subgroups or 2-maximal subgroups have the cover-avoiding properties. Recently, in [4], Fan, Guo and Shum generalized the cover-avoiding properties of finite groups. They called Semi cover-avoiding properties, that is, a subgroup H is said to be semi cover-avoiding in a group G if there is a chief series 1=G_0 \leq G_1 \leq \ldots \leq G_m =G, such that for every i=1,\ldots ,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}. They used Semi cover-avoiding properties of Sylow and maximal subgroups to investigate the solvability of finite groups.

In this paper, we will push further this approach and obtain the supersolvability for a finite group based on the assumption that minimal subgroups and cyclic subgroups of order 4 have the semi cover-avoiding properties. Our main result is the following:

Theorem A. Let G be a finite group. N \lhd G and G/N is supersolvable. If every minimal subgroup and every cyclic subgroup of order 4 of N are semi cover-avoiding subgroups of G, then G is supersolvable.

2. Basic definitions and preliminary results
In this section, we give one definition and some lemmas which are useful for our main results.

Definition 2.1 ([4]) A subgroup H is said to be semi cover-avoiding in a group G if there is a chief series 1=G_0 \leq G_1 \leq \ldots \leq G_m =G, such that for every i=1,\ldots ,m either H covers G_i/G_{i-1} or H avoids G_i/G_{i-1}.
The following Lemma is obvious.

**Lemma 2.2** Let H be a subgroup of a group G and let \(1 \leq \ldots \leq G_N\) be a normal series of G. If the subgroup H covers (respectively avoids) \(M/N\), then H covers (respectively avoids) any quotient factor between \(N\) and \(M\) of any refinement of the normal series.

**Lemma 2.3** Let G be a group. Let H be a semi cover-avoiding subgroup of G and let N be a normal subgroup of G.

1. If \(H \leq K \leq G\), then H is a semi cover-avoiding subgroup of K.
2. If \(N = H\) or \(\gcd(|H|, |N|) = 1\), where \(\gcd(-, -)\) denotes the greatest common divisor, then HN/N is a semi cover-avoiding subgroup of G/N.

**Proof.** (i) Since H is a semi cover-avoiding subgroup of G, that is, there exists a chief series

\[1 = G_0 \leq G_1 \leq \ldots \leq G_m = G\]

such that for every \(i = 1, \ldots, m\) either H covers \(G_i / G_{i-1}\) or H avoids \(G_i / G_{i-1}\). Suppose that \(K_i = G_i \cap K\), \(i = 1, \ldots, m\). Then

\[HK_i = K_i \cap H\] and \(H \cap K_i = K_i \cap H\).

So H semi covers or avoids a normal series

\[1 = K_n \leq K_{n-1} \leq \ldots \leq K_0 = K\].

By Lemma 2.2 we know, H is a semi cover-avoiding subgroup of K.

(ii) By Lemma 2.2 in [4], we can prove the result.

**Lemma 2.4** ([6]) Suppose that G is an inner-supersolvable group, then

1. there exists a normal \(P \leq \text{Syl}_p(G)\) such that \(G \leq P \times M\). \(P/\Phi(P)\) is a minimal normal subgroup of G/\(\Phi(P)\).
2. if \(p > 2\), then \(\exp(P) = 2\); if \(p = 2\), then \(\exp(P) \leq 4\) and \(p^2 \parallel |G|\).
3. if P is Abelian, then \(\Phi(P) = 1\).
4. if P is not Abelian, then \(\Phi(P) = Z(P) = P/\Phi(P)\).
5. G is a group with Sylow tower and G is an inner-nilpotent group.

3. **Proof of the Main Theorem**

**Proof of Theorem A.** Assume that the theorem is false and let G be a counterexample of minimal order.

- G is an inner-supersolvable group.
- In fact, for any proper subgroup H of G, we have that HN/N \(\equiv H \cap H \cap N\). Since G/N is supersolvable, we know that H/N is supersolvable. By Lemma 2.3 we know that every minimal subgroup and every cyclic subgroup of order 4 are semi cover-avoiding subgroups of H. So H is supersolvable. That is, G is an inner-supersolvable group. By Lemma 2.4 we can get that there exists a normal \(P \leq \text{Syl}_p(G)\) such that \(G = P \times M\). \(P/\Phi(P)\) is a minimal normal subgroup of G/\(\Phi(P)\).

References