International Journal of Statistics and Applied Mathematics 2017; 2(2): 55-57

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2017; 2(2): 55-57

www.mathsjournal.com
Received: 19-01-2017
Accepted: 22-02-2017

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Set Theory: Its Operations, Types and Applications

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Abstract
Set theory is a branch of mathematical logic that studies about sets and various operations performed on sets. In this paper, binary operations on sets are defined and different types of sets are also discussed. Basic properties of sets are also highlighted in this paper. There is a detailed description of applications of set theory. Different areas of study of set theory are also mentioned thoroughly in this paper.

Keywords: Set theory, mathematical logic, operations, Applications

Introduction

Set theory is a branch of mathematical logic that studies about sets and various operations performed on sets. A set is a well-defined collection of objects. We can collect any type of objects in a set, but set theory is applicable to objects that are relevant to mathematics. The language of set theory is used to define almost all the mathematical objects. Set theory starts with a fundamental binary relation between an object o of any set A and the set A itself. If o is an element of set A, the mathematical expression for this is: $o \in A$ is used. Since sets are made up of objects, hence the membership relation can relate sets also.

Set Inclusion: It is a subset and derived binary relation between two sets. If all the members of set A are also members of set B, then A is a subset of B, it is denoted by $A \subseteq B$. For example, {1, 2, 3, 4} is a subset of {1, 2, 3, 4, 5}, and {5} is also a subset of B but {1, 6} is not a subset of set B. Any set is also a subset of itself. For cases where this chance is unsuitable or would make sense that it can be rejected, the term proper subset is defined. A is called a proper subset of B if and only if (iff) A is a subset of B, and A is not equal to B. Also let us consider a set A= {1, 2, 3}; then 1, 2, and 3 are elements of the set A but are not subsets of A whereas the subsets, such as {1}, {1, 3} etc. are not the elements of the set A{1, 2, 3}.

Binary operations on sets: Set theory features binary operations on sets like arithmetic numbers. These operations are:

- Union: Union of the sets A and B is denoted by $A \cup B$. It is the set of all elements that are a member of either set A, or set B, or both sets. The union of {1, 2, 4} and {1, 3, 4} is the set {1, 2, 3, 4}.
- Intersection: Intersection of the sets A and B is denoted $A \cap B$. It is the set of all objects that are members of both the sets A and B. The intersection of {1, 2, 4} and {1, 3, 4} is the set {1, 4}.
- Set difference: Set difference of U and A is denoted by $U \setminus A$ or U - A. It is the set of all elements of U that are not members of A. The set difference {1, 2, 3, 4} \ {2, 3, 4, 5} is {1}, on the other hand, its converse, the set difference {2, 3, 4, 5} \ {1, 2, 3, 5} is {4}. When the set A is a subset of U, then the set difference $U \setminus A$ is also known as the complement of A in U. Thus, if the choice of U is clear from the context, the notation A^c is also used instead of $U \setminus A$, particularly in case, if U is a universal set as we study in Venn diagrams.
- Symmetric difference: Symmetric difference of sets A and B is denoted $A \vartriangle B$ or $A \ominus B$. It is the set of all elements that are a member of exactly one of the set A and B (i.e. elements which are in any one of the two sets, but not in both sets). E.g., for the sets A= {1, 2, 3, 4} and B= {2, 3, 4, 5}, the symmetric difference set is {1, 5}.

It is also defined as the set difference of the union and the intersection of sets A and B, i.e. (A ∪ B) \ (A ∩ B) or (A - B) ∪ (B - A) or (A \ B) ∪ (B \ A).

- Cartesian product: Cartesian product of A and B is denoted A × B. It is the set whose elements are all possible ordered pairs (a, b) where a is the member of A and b is the member of B. The Cartesian product of {1, 2, 3} and {a, b} is {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}.
- Power set: Power set of any set A is the set whose elements are all of the possible subsets of A. For example, the power set of A= {1, 2, 3} is P(A)= { {}, {1}, {2}, {3} {1, 2},{2, 3},{1, 3},{1, 2, 3} }. The number of elements in power set P(A) is 2^n, where n is the number of elements in set A.

Types of sets: There are many types of sets. These are

Empty Set or Null Set: A set which does not possess any element, i.e., there is no element in set, is called an empty set. It is also known as the null set or the void set. It is denoted by ∅ (phi) in set builder form. In roster form, ∅ is denoted by {}. It is a finite set because the number of elements in an empty set is 0 and 0 is finite.

For example

Set of natural numbers less than 1. We know that there is no natural number less than 1. Therefore, it is an empty set.
Let A = {n: n is a composite number smaller than 4}. Here A is an empty set because there is no composite number less than 4.
We also know that ∅ has no element. Thus, ∅ ≠ {0}, because {0} is a set which has one element 0.
Cardinal number is the number of elements present in a set. Hence, the cardinal number of an empty set, i.e., n (∅) = 0.
Singleton Set: It is a set which consists of only one element.

For example

A = {z: z is neither a prime number nor a composite number}. It is a singleton set containing one element, i.e., 1, because 1 is neither prime nor composite.
B = {k: k is a natural number, k <= 1}. This set contains only one element '1' because there is no natural number less than 1 and hence it is a singleton set.
C = {n: n is an even prime number}. Here C is a singleton set because 2 is the only prime number which is even.

Finite Set: It is a set which contains a definite (fixed) number of elements. Empty set is also called a finite set because it contains no element, i.e. 0 elements.

For example

• The set of all colours in the rainbow. It contains 7 colours.
• N = {n: n ∈ N, 3< n < 7}
• Set of prime numbers less than 37, P = {2, 3, 5, 7, 11,…….., 31}

Infinite Set: It is the set whose elements cannot be listed, i.e., it is the set which contains never-ending elements.
For example: Set of all the points on an axis.
A = {x: x ∈ Z, x <1}
Set of all even numbers.
We should know that all infinite sets can't be expressed in roster form.

For example

The set of rational numbers since the elements of this set do not follow any particular pattern.
Cardinal Number of a Set: It is the number of distinct elements in a given set A. It is represented as n(A).

For example

A {n: n ∈ N, n <6}, i.e. A = {1, 2, 3, 4, 5}
Therefore, n (A) = 5.
B = set of alphabets in the word GEOMETRY. Thus, B = {G, E, O, M, T, R, Y}.
Therefore, n (B) = 7. It must be taken into knowledge that repeated elements must be taken only one time.
Equivalent Sets: Two sets P and Q are said to be equivalent if they have same number of elements, i.e. their cardinal number is same, i.e., n (P) = n (Q). The equivalent set is denoted by the symbol '↔'.
For example: P = {a, b, c} and Q = {2, 5, 8}, Here n (P) = 3 and n (Q) = 3. So,
P ↔ Q.

Equal sets: Two sets A and B are said to be equal if they have all the same elements. Every element of A is an element of B and every element of B is an element of A.

For example: A = {a, b, c, d} and B = {d, a, c, b}. Therefore, A = B

Properties of sets: The two basic properties of set are:

1. If we change the order of writing the elements, then it does not make any changes in the set, i.e. order of writing the elements is not important.
   Thus, the set {p, q, r} can also be written as {p, r, q} or {r, p, q} or {q, r, p} etc.
   E.g. Set A = {1, 2, 3, 4} is same as set A = {4, 2, 1, 3}
2. If one or more elements of a set are repeated, then also the set remains the same, i.e. all the elements of a set should be different. So, if any element of a set is repeated, it is considered as a single element only. Thus, {1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 4} = {1, 2, 3, 4}

The set of letters in the word 'Notebook' = {N, O, T, E, B, K}
In general, the elements of a set are not repeated. Thus,
Set of vowels used in the word 'Audition' is Set V={A, U, I, O}

Applications: Many mathematical concepts can be defined easily by using the concepts of set theory. For example, mathematical structures such as graphs, manifolds, rings, and vector spaces can be defined as sets satisfying various axioms or properties. Equivalence and order relations are present in mathematics. The theory of mathematical relations can also be described in set theory.
Set theory is also a promising foundational system for a lot of mathematics. Most of the mathematical theorems can be derived using an appropriately designed set of axioms for set theory, augmented with many definitions, using first or second order logic. For example, properties of the natural and real numbers can be derived by using set theory, as each number system can be identified with a set of equivalence classes under a suitable equivalence relation whose field is some infinite set.
Set theory is a base for mathematical analysis, topology, abstract algebra, and discrete mathematics. Mathematicians

accept that the theorems in these areas can be derived from the appropriate definitions and the axioms of set theory. Few full derivations of complex mathematical theorems from set theory have been formally verified, however, because such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, Metamath, includes human-written, computer□verified derivations of more than 12,000 theorems starting from ZFC set theory, first order logic and propositional logic.

Area of study of set theory: Set theory is a wider area of research in mathematics and interrelated subfields.

Combinatorial set theory: It concerns extensions of finite combinatorics to infinite sets. This includes the study of cardinal arithmetic and the study of extensions of Ramsey's theorem. E.g. Erdős–Rado theorem.

Descriptive set theory: It is the study of subsets of the real line and, subsets of Polish spaces. It starts with the study of point classes in the Borel hierarchy and then extends to more complex hierarchies like the projective hierarchy and the Wadge hierarchy. Many properties of Borel sets can be confirmed in ZFC, but proving these properties hold for more complicated sets. It requires additional axioms related to determinacy and large cardinals.

The field of effective descriptive set theory lies between set theory and recursion theory. It includes the study of lightface point classes. It is closely related to hyper arithmetical theory. In most of the cases, results of classical descriptive set theory have effective versions; while in some other cases, new results are obtained by first proving the effective version and then extending it to make it applicable widely.

Fuzzy set theory: In set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an element is either a member of a set or not. In this theory, this condition was relaxed, so an element has a degree of membership in a set, a number between 0 and 1. For example, in the set of "younger people", the degree of membership of a person represented by real number 0.82 is more flexible than a simple answer in yes or no.

Other areas of study of set theory are

Inner model theory: It is a transitive class set theory. It includes all the ordinals and satisfies all the axioms of ZF. The constructible universe L developed by Gödel is its example. It is used to prove consistency results.

Large cardinals: It is a cardinal number with some extra properties. These include inaccessible cardinals, measurable cardinals, and many more. These properties imply that the cardinal number should be very large.

Set-theoretic topology: It studies general topology that are set-theoretic in nature or that require advanced methods of set theory to solve the problems. Many theorems are independent of ZFC. They require strong axioms for their proof. E.g. normal Moore space problem in general topology was a subject of intense research. The answer to this question was eventually proved to be independent of ZFC.

Determinacy: It is a fact that under applicable assumptions, a two-player games of perfect information are examined from the start means that one player must have a winning strategy. The existence of these strategies has necessary consequences in descriptive set theory. The axiom of determinacy (AD) is a key object to study; although incompatible with the axiom of selection, AD implies that all subsets of the real line are well behaved. AD is also used to prove that the Wadge degrees having elegant structure.

Forcing: Paul Cohen invented the method of forcing when he was searching for a model of ZFC in which the continuum hypothesis fails, or a model of ZF in which the chosen axiom fails. Forcing adjoins to some model of set theory additional sets in order to create a larger model with properties determined (i.e. "forced") by the construction and the original model. Forcing is a method for proving relative consistency by finitistic methods.

Cardinal invariants: It is the real line property measured by a cardinal number. For example, a well-studied invariant is the smallest cardinality of a collection of meagre sets of real numbers whose union is the complete real line. These are invariants means that any two isomorphic models of set theory must give the same cardinal for each invariant. The relationships between the cardinal invariants are often complex and related to axioms of set theory.

Conclusion

Set theory is a branch of mathematical logic that studies about sets and various operations performed on sets. A set is defined as a well-defined collection of objects. Binary operations on sets like union intersection, difference, Cartesian product etc. are defined. Many types of sets like empty set, finite set, equal sets etc. are also discussed here briefly. Basic properties of sets are also explained in this paper. There is a detailed description of applications of set theory. Different areas of study of set theory like combinatorial set theory, forcing, cardinal invariants etc. are also mentioned.

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