Almost periodic points and minimal sets in topological spaces

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Abstract
This paper we characterize in this paper, we need the following terminology and concepts. Let \( N = \{0,1,2,\ldots\} \) be the set of natural numbers and let \( Z = \{0, \pm 2, \ldots\} \) be the set of integers. For a set \( A, |A| \) denotes the cardinality of the set \( A \). If \( f : X \to X \) is a continuous function of a topological space \( X \), then \( f_0 = \text{Id} \) \( (n \geq 1) \) denotes the composition with itself \( n \) times. The orbit of a point \( x \in X \) under \( f \), denoted by \( O(x, f) \), is the set \( \{f^n(x) \mid n \in \mathbb{N}\} \). Also if \( f : X \to X \) is a homeomorphism, then we put \( f^{-n} (f^{-1}) = n (n \geq 1) \), where \( f^{-1} \) is the inverse of \( f \). The two-sided orbit of a point \( x \in X \) under \( f \), denoted by \( O(x, f) \), is the set \( \{f^n(x) \mid n \in \mathbb{Z}\} \). A point \( x \in X \) is called a periodic point of \( f \) if there exists a positive number \( N \in \mathbb{N} \) such that \( f^n(x) = x \). A point \( x \in X \) is called an almost periodic point of \( f \) provided that for any neighborhood \( U \) of \( x \) in \( X \), there exists \( N \in \mathbb{N} \) such that \( \{f^n(x) \mid n \in \mathbb{Z}\} \). We denote the set of all almost periodic points of \( f \) by \( \text{AP}(f) \). A subjet of \( X \) is a compact minimal set if \( W \) is a closed invariant set of \( f \) and \( W \) does not contain any proper closed invariant set. A map \( f : X \to X \) is minimal if \( X \) is a minimal set. It is well known that if \( f : S^1 \to S^1 \) is an irrational rotation of the unit circle \( S^1 \), then \( f \) is minimal.

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Introduction
A topological space \( X \) is a \( T_1 \)-space if for any distinct point \( x \) and \( y \) in \( X \), there exist open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( x \notin V \). A topological space \( X \) is Hausdorff space if for any distinct points \( x \) and \( y \) in \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \). A topological space \( X \) is regular space if for any closed subset \( W \) of \( X \), any point \( x \in X \) and \( W \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( W \subseteq V \). A topological space \( X \) is an \( ! \)-regular space if for any closed subset \( W \) of \( X \), any point \( x \in X \) and \( W \), and any countable subset \( A \) of \( W \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( A \subseteq V \) [1, 2, 3].

Now we assert and prove the main theorem on compact minimal set in topological dynamics.

Theorem:-
Let \( E \) be a compact minimal set in \( X \). The set \( E \) is almost periodic minimal if and only if the flow \( \pi \) is equicontinuous on \( E \).

Proof:-
Let us assume \( E \) be almost periodic minimal set and let \( \pi = H(x) \), where \( x \) is and almost periodic point. We claim that the flow \( \pi \) is equicontinuous on \( E \).

We first show that \( \pi \) is equicontinuous on \( y(x) \). Let an index \( a \in A \) be preassigned and select \( b \in A \), so by the uniformity condition if follows that \( X_i \in V_b(x_i+1), i = 1,2,3 \)

\[ x_i \in V_b(x_i) \quad \ldots \quad \ldots \quad \ldots \quad (1) \]

Let \( L \) be an inclusion interval for the relatively dense set \( E (b, \pi x) \).

Since the continuity of \( \pi \) restricted to the compact set \( Ex \) is uniform, for every \( b \in A \) there exists and index \( c \) such that for \( y, z \in y \in V_c(z) \) implies that,

\[ \pi(y, t) \in V_b(\pi(z, t)) \quad \text{for all} \quad t \in R \quad \ldots \quad \ldots \quad (2) \]
In order to show that $\pi$ is equicontinuous on $\gamma(x)$ for any tow points $\pi(x, t_1)$ and $\pi(x, t_2)$ of $\gamma(x), \pi(x, t_1 + t) \in V_2(\pi(x, t_2 + t))$ whenever $\pi(x, t_1 + t) \in V_2(\pi(x, t_2 + t))$ for all $t \in \mathbb{R}.$ Let $t \in \mathbb{R}$ be fixed. Then there exists a $T \in E(b, \pi x)$ such that $0 \leq t + r \leq L.$ From the definition of $b$-displacement of $\pi x,$ it follows that, $\pi(x, t + t_1) \in V_1(\pi(x, t_1 + t))$ and $\pi(x, t + t_1 + t) \in V_2(\pi(x, t_2 + t))$

Applying the condition (2) of uniform continuity one has $\pi(x, t_1 + t) \in V_2(\pi(x, t_2 + t))$ whenever $\pi(x, t_1) \in V_2(\pi(x, t_2))$ i.e., the equicontinuity of $\pi$ on $\gamma(x)$ follows.

In order to show that $\pi$ is equicontinuous on $E,$ let $a \in A$ be any index. Then there $b \in A$ in consistent with the uniformity condition (i). By using the equicontinuity of $\pi$ on $\gamma(x)$ we can choose $C \in A$ such that $\pi(x, t_1 + t) \in V_a(\pi(x, t_2 + t))$ whenever $\pi(x, t_1) \in V_a(\pi(x, t_2))$

Now let us choose d so that $x_i \in V_a(x_i) + \epsilon_i, i = 1, 2, 3$

We will now show that if $y, z$ are points in $E = H(x)$ with $y \in V_2(z),$ then $\pi^n(y) \in V_3(\pi^n(z)),$ i.e., $\pi(y, t) \in V_3(\pi(z, t)).$

Since $y, z \in E = H(x),$ there are nets $\{\pi(x, t_n)\}$ and $\pi(x, S_n)$ in $\gamma(x)$ with $\pi(x, t_n) \to y$ and $\pi(x, S_n) \to z$ without the loss of generality. $\pi(x, t_n) \in V_2(y)$ and $\pi(x, S_n) \in V_2(z).

Since $y \in V_2(z) \Rightarrow \pi(x, S_n + t) \in V_2(\pi(x, t_n))$ for all $n$ all $t \in \mathbb{R}.$ Thus we have two convergent nets $\{\pi(x, S_n + t)\}$ and $\{\pi(x, S_n + t)\}$ in the compact hull $H(x) = E,$ so they are uniformly convergent.

Now for the index $b \in A,$ there exists no such that $\pi(x, S_n + t) \in V_2(x, t) \text{and} \pi(x, t_n + t) \in V_2(y, t)$ for all $n \geq m_0$ and $t \in R.$ Finally, for the uniformity condition (1) where $x_1 = \pi(y, t)x_2 = \pi(x, t_1 + t)$ and $x_3 = \pi(x, S_n + t)$ and $x_4 = \pi(z, t),$ it follows that $\pi(y, t) \in V_3(\pi(z, t)).$

Conversely, let us assume that the flow $\pi$ is equicontinuous on $E.$ We show that $E$ is almost periodic minimal. Since $E$ is compact minimal, then by Birkhoff recurrence theorem every motion in $E$ is recurrent. Therefore, for any $x \in E$ and any index by the set $E(b, \pi x) = \{t: \pi(x, t) \in V_2(x)\}$ is relatively dense in $E.$ Let the index $a$ be preassigned, then the resists $b \in A$ for the equicontinuity of $\pi.$ Thus we have $E(b, \pi x, t) \in V_2(x)$ for all $t \in R.$

Assuring the almost periodicity of $\pi x.$ Therefore, $E$ is almost periodic minimal.

References