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Common fixed point theorems in C* -Algebra-valued b-metric spaces

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Abstract
By introducing the concept of the contraction of one mapping with respect to another mapping in a C* -algebra-valued b-metric spaces, the paper gives the coincidence and common fixed point of the two mapping in a C* -Algebra-valued b-metric spaces, contracting mapping, coincidence point and common fixed point.

Keywords: one mapping, C* -Algebra-valued b-metric spaces

Introduction and Preliminaries

Banach contraction principle is a classical and celebrated result of fixed point theory in metric space. Its application is obvious seen in applied mathematics and physics.

In 1993, the axiom for semi metric spaces, was put forth by Czerwik [1] generalizing the Banach contraction principle. Thereafter, many papers have dealt with fixed point theory in such space [2, 5]. The relaxation of the triangle of the triangle inequality is also discussed in Fagin and Stock Meyer [6], who call this new distance measure nonlinear elastic matching.

Xia [9], the authors introduced the concept of C* - algebra-valued metric spaces. The main idea consists in using the set of all positive elements of a unital C* -algebra instead of the set of real numbers. Obviously such spaces generalize the concept of metric spaces. In this paper, as generalization of b-metric spaces and operator- valued metric spaces [12], we introduce a new type of metric spaces, namely, C* - algebra-valued b-metric spaces, and give some fixed point theorems for self-map with contractive condition on such spaces.

To begin with, we collect some definitions and basic facts on the theory of C* -algebras, which will be need in the sequel. Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate- linear map a →a* on A such that a** = a and (ab)* = b*a* for all a, b ∈ $\mathbb{A}$. The pair $(\mathbb{A},*)$ is called a$*$ -algebra [13, 14]. A $*$ Banach -algebra is a $*$ - algebra $\mathbb{A}$ together with a complete sub multiplicative norm such that $\|a^*a\| = \|a\|^2$ (∀a∈ $\mathbb{A}$).

Notice that the seeming mild requirement on a C* - algebra above is in fact very strong. It is clear that under the norm topology, L(H), the set of all bounded linear operators on a Hilbert space H, is a C*- algebra. Furthermore, give a C* - algebra $\mathbb{A}$. there exists a Hilbert space H and a faithfully $*$- representation (π, H) of A such that $\pi(\mathbb{A})$ can be made a closed C* - sub algebra of L (H) [13].

Throughout this paper, by $\mathbb{A}$ we always denote an unital C*- algebra with a unit I. Set $\mathbb{A}_h = \{\times \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \geq 0$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset (0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define partial ordering $\leq$ on x $\mathbb{A}_h$ as follows: $x \leq$ y and only if {y - x ≥0} and $\| x \| = (x^* x)^{1/2}$.

Lemma 1.1 [13, 15] suppose that $\mathbb{A}$ is a unital C*-algebra with a unital I.

(1) For any $x \in \mathbb{A}_+$ we have $x \leq I \Leftrightarrow \| x \| \leq 1$.

(2) If a∈ $\mathbb{A}_+ = \|a\| < \frac{1}{2}$ then I-a is invertible and $\|a(I-a)^{-1}\| < 1$

(3) Suppose that a,b ∈ $\mathbb{A}$ with a,b ≥ 0 and ab =ba then ab ≥ 0.

(4) By $\mathbb{A}$ we denote the set {a ∈ A:ab = ba, $\forall b \in \mathbb{A}$}. Let $a \in \mathbb{A}$, if $b, c \in \mathbb{A}$ with $b \geq c \geq 0$, and $I - a \in \mathbb{A}_+$ is an invertible 0perator, then

$$(I-a)^{-1}b \leq (I-a)^{-1}c$$

With the help of the positive elements in $\mathbb{A}$ one can gives the definition a C*- algebra-valued b-metric space.
Definition 1.1 [7] Let X be a nonempty set and $A \in \mathbb{A}$ such that $A \geq I$ Suppose the mapping $d: X \times X \to \mathbb{A}$ satisfies:
1. $0 \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0 \Leftrightarrow x = y$;
2. $d(x,y) = d(y,x)$ for all $x, y \in X$;
3. $d(x,y) \leq A[d(x,z) + d(z,y)\ for\ all\ x, y \in X.$

Then d is called a C* algebra-valued b-metric on X and (X, $\mathbb{A}, d$) is a C*-algebra-valued b-metric space.
Definition 1.2 [7] Let (X, $\mathbb{A}, d$) be a C* algebra valued b-metric space. Suppose that-$\{x_n\} \subset X$ and $x \in X$. If for any $\epsilon > 0$, there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \epsilon$, then$\{x_n\}$ tis said to be converge with respect to $\mathbb{A}$ and converges to x and x the limit of $\{x_n\}$. We denote it by him-
If for any $\epsilon > 0$ ther e is N such that for all n, m >N, $\|d(x_n, x_m)\| \leq \epsilon$ then $\{x_n\}$ is called a Cauchy sequence with respect to.
We say (X, $\mathbb{A}, d$)is a complete C* algebra valued be metric space if every Cauchy sequence with respect to $\mathbb{A}$ is converge.
Remark 1.1 If $\mathbb{A} = 1$ then the ordinary triangle inequality in a C* -algebra-valued metric space satisfied. Thus a C* -algebra-valued b-metric space is an ordinary C*-algebra-valued metric space. In particular, if $\mathbb{A} = C$ and $\mathbb{A}$ = 1, the C*-algebra-valued b-metric spaced are just the ordinary metric space. The following example illustrates that, in general, a $C^* -$ algebra valued metric space, one can see [11].
Example 1.1 Let X = $\mathbb{A}$ and A = M_n (R). Define d (x,y) = diag (c_1 |x-y|^{p,} c_2 |x-y|^{p,} C_2 |x-y|^{p,} …., C_n |x-y|^p) with 'diag' denotes a diagonal matrix and x,y ϵ R, C_i > 0 (i = 1,2,…,n) are constants and p>1. It is easy to verify that d (x,y) is a complete C*-algebra-valued b-metric, for proving (3) of Definition 1.1 we only need to use the following inequality:

$$|x-y|^p \leq 2(|x-z|^p + |z-y|^p),$$

Which implies that d(x,y) ≤ A [d (x,z)+ d (z,y)] for all x,y,z ∈ X, where A = X, where A = $2^p I$ ∈ A' and A> I by 2^p > 1. But |x-y|^p + |z-y|^p is impossible for all x > z> y [16]. Thus (X, M_n (R), d) is not a C*-algebra-valued metric space.
Definition 1.3 [7] suppose that (X, A, d) is a C*-algebra-valued be-metric space. We call a mapping *T:* X → X a C*-algebra-valued contractive mapping on X, if there exists a B ∈ A with ||B || < 1 such that d (Tx, Ty) ≤ B* d (x, y) B, ∀ x, y ∈ X.
Definition 1.4 [8] suppose that (X, A, d) is a C*-algebra-valued b-metric space and T, f: X → X be the self-mappings. A point x in X is called a coincidence point (common fixed point) if Tx = fx (Tx = fx = x). Also the pair of mappings T, f: X → X are compativel if they commute on the set of coincidence points.
Theorem 1.1 [7] If (X, A, d) is a complete C*-algebra-valued b-metric space and T: X→X is a contractive mapping, there exists a unique fixed point in X.

Main Result
First, we give the definition of a contractive mapping with respect to another mapping.

Definition 2.1 Suppose that (X, A, d) is a C*-algebra-valued b-metric space. We call a mapping T: X → X a C* -algebra-valued contractive mapping on X, if there exists a $B \in\_ \mathbb{A}$ with $\|B\| \leq 1$ such that $d(Tx, Ty) \leq B^* d(fx, fy)B$

Theorem 2.1 If (X, A, d) is a C* algebra-valued b-metric space and T: X → X is a contractive mapping with respect to mapping f on X. If $T(X) \subseteq f(X)$ and $f(X)$ is complete subspace of X. Then T and f have coincidence point. Further if T and f are compatible, then there exists a unique fixed point of T and f in X.
Proof: It is clear that if B=0 T maps the X into a single point. Thus without loss of generality, one can suppose that $B \neq 0$. Let x_0 be any arbitrary point in X. Choose a point x_1 in X such that $Tx_0 = f(x_1)$. It is possible due to $T(X) \subseteq f(X)$. Continuing this process indefinitely, for each n in X, one can find a x_{n+1} such that $y_n = Tx_n = f(x_{n+1})$. Without loss of generality, one may assume that $y_{n+1} \neq y_n$for all n in N, otherwise f ande T have coincidence point. For convenience, by B_0 we denote the element $d(x_1, x_1)$ in $\mathbb{A}$. In case$y_{n+1} \neq y_n$ we have
Notice that in a C* -algebra, if $a, b \in \mathbb{A}_+$ and $a \leq b$, then for any $x \in A$ both x^*ax and x^*bx are positive elements and $x^*ax \leq x^*bx$[13]. Thus

$$\begin{aligned} d(y_{n+1}, y_n) &= d(Tx_{n+1}, Tx_n) \\ &\leq B^* d(fx_{n+1}, fx_n)B \\ &= B^* d(y_n, y_{n-1})B \\ &\leq (B^*)^2 d(y_n, y_{n-1})B^2 \end{aligned}$$

Continuing this way, we get

$$\begin{aligned} &\leq (B^*)^n d(y_n, y_{n-1})B^n \\ &= (B^*)^n B_0 B^n \end{aligned}$$

For any m≥ 1, it follows that

$$d(y_{n+1}, y_n) \leq Ad\left(y_{m+p}, y_{m+p-1}\right) + Ad\left(y_{m+p-1}, y_m\right)$$

$$\le Ad(y_{m+p}, y_{m+p-1}) + A^2 d(y_{m+p-1}, y_{m+p-2}) + A^2 d(y_{m+p-2}, y_m)$$
$$\le Ad(y_{m+p}, y_{m+p-1}) + A^2 d(y_{m+p-1}, y_{m+p-2}) + \cdots$$
$$+A^{p-2} d(y_{m+2}, y_{m+1}) + A^{p-1} d(y_{m+1}, y_m)$$
$$\le A(B^*)^{m+p-1} B_0 B^{m+p-1} + A^2 (B^*)^{m+p-2} B_0 B^{m+p-2}$$
$$+A^3 (B^*)^{m+p-3} B_0 B^{m+p-3} + \cdots + A^{p-1} (B^*)^m B_0 B^m$$
$$\sum_{k=1}^{p-1} A^k (B^*)^{m+p-k} B_0 B^{m+p-k} +$$
$$A^{p-1}(B^*)^m B_0 B^m$$
$$\sum_{k=1}^{p-1} \left(B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\right)^* \left(B_0^{\frac{1}{2}} A^{\frac{k}{2}} B^{m+p-k}\right) +$$
$$\left(B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\right)^* \left(B_0^{\frac{1}{2}} A^{\frac{p-1}{2}} B^m\right)$$
$$\to 0 (m \to \infty)$$

Therefore $\{y_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $f(X)$, there exists an $y \in f(X)$ such that $lim_{n\to\infty} y_n = lim_{n\to\infty} Tx_n = lim_{n\to\infty} fx_{n+1} = y$. Consequently, we obtain a point p such that $fp = y$. Now we show that p is the coincidence point of f and T.

Since
$$0 \le d(Tp, fp) \le AB^* d(fp, fx_n)B + d(y_{n+1}, y) \to 0 \ (n \to \infty)$$
Hence,
$$Tp = fp.$$
Further suppose
$$Tp = fp = q.$$

Since f and T are compatible, therefore $fTp = Tfp$ or $Tq = fq$.
Now we show that q is the fixed point of f and T.
Now
$$0 \le d(fq, q) = d(Tp, Tp)$$
$$\le B^* d(fq, fp)B$$
$$= B^* d(fq, q)B$$
This implies
$$0 \le \|d(fq, q)\| \le \|B^* d(fq, q)B\|$$
$$< \|d(fq, q)\|$$

which is impossible. Hence q is fixed point of f and$T$.
Now suppose that x is another fixed point of T and f other than q.
Now since

$$0 \le d(q, x) = d(Tq, Tx)$$
$$\le B^* d(fq, fx)B.$$
We have
$$0 \le \|d(q, x)\| \le \|B^* d(fq, fx)B\|$$
$$< \|d(q, x)\|$$
It is impossible. So fixed point is unique.

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