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## Observation on $y^2 = 6x^2 + 1$

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**Abstract**

The binary quadratic equation  $y^2 = 6x^2 + 1$  is analyzed for its distinct integer solutions and we obtain infinitely many Pythagorean triangles. A few interesting relations among the sides are also given.

**Keywords:** The binary quadratic equation, Pell equation, Pythagorean triangles, integer solutions

**1. Introduction**

The Diophantine equation of the form  $x^2 = Dy^2 + 1$  where  $D$  is a given positive nonsquare integer and integer solutions sought for  $x$  and  $y$ . This equation was first studied in India, starting with Brahmagupta, who developed the Chakravala method to solve Pell's equation. Pell's equation has infinitely many distinct integer solutions when  $D$  takes different numerical values [1-8]. In this communication, by employing the non-trivial integral solutions of the binary quadratic equation  $y^2 = 6x^2 + 1$ , we obtain infinitely many Pythagorean triangles. The recurrence relations satisfied by the sides of the triangle are presented. Also, A few interesting relations among the sides are obtained.

**2. Method of Analysis**

Consider the binary quadratic equation

$$y^2 = 6x^2 + 1$$

whose general solution  $(x_n, y_n)$  is represented by

$$x_n = \frac{1}{2\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1} \right]$$

$$y_n = \frac{1}{2} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right], \quad n = 0, 1, 2, 3, \dots$$

Assuming  $x_n = b_n$  and  $y_n = a_n - \frac{5}{2}b_n$ , we obtain

$$a_n = \frac{1}{4\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+2} - (5 - 2\sqrt{6})^{n+2} \right]$$

$$b_n = \frac{1}{2\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1} \right], \text{ where } n = 0, 1, 2, 3, \dots$$

Considering  $a_n, b_n$  as the generators of a Pythagorean triangle, its legs  $X_n, Y_n$  and hypotenuse  $Z_n$  are found to be

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$$X_n = 2a_n b_n = \frac{1}{24} \left[ (5 + 2\sqrt{6})^{2n} (485 + 198\sqrt{6}) + (5 - 2\sqrt{6})^{2n} (485 - 198\sqrt{6}) - 10 \right]$$

$$Y_n = \frac{1}{96} \left[ (5 + 2\sqrt{6})^{2n} (4605 + 1880\sqrt{6}) + (5 - 2\sqrt{6})^{2n} (4605 - 1880\sqrt{6}) + 6 \right]$$

$$Z_n = \frac{1}{96} \left[ (5 + 2\sqrt{6})^{2n} (4997 + 2040\sqrt{6}) + (5 - 2\sqrt{6})^{2n} (4997 - 2040\sqrt{6}) - 10 \right]$$

The above values of  $X_n, Y_n$  and  $Z_n$  satisfy the following recurrence relations:

$$X_{n+2} - 98X_{n+1} + X_n = 40$$

$$Y_{n+2} - 98Y_{n+1} + Y_n = -6$$

$$Z_{n+2} - 98Z_{n+1} + Z_n = 10$$

The following table shows a few numerical Pythagorean triangles:

$n$	$X_n$	$Y_n$	$Z_n$
0	40	96	104
1	3960	9401	10201
2	388080	921196	999604

**2.1 Properties**

We present a few interesting relations among the sides in each of the above Pythagorean triangles.

- $X_n \equiv 0 \pmod{40}$ .
- $P_n$  denotes the perimeter of the Pythagorean triangle and we obtain

$$P_n \equiv 0 \pmod{2}.$$

- $P_{2n}$  can be expressed as the difference of two squares.
- $Y_{2n} \equiv 0 \pmod{2}$ .
- $Z_{2n} \equiv 0 \pmod{2}$ .
- $A_n$  denotes the Area of the Pythagorean triangle and we obtain

$$A_n \equiv 0 \pmod{10}.$$

- From each values of  $X_n (n = 0, 1, 2, \dots)$ , one can generate second order Ramanujan Numbers.

For example

$$40 = 2 \times 20 = 4 \times 10 = 11^2 - 9^2 = 7^2 - 3^2.$$

$$11^2 - 9^2 = 7^2 - 3^2 \Rightarrow 11^2 + 3^2 = 7^2 + 9^2 = 130.$$

Here we observe that 130 is second order Ramanujan Number.

**3. Conclusion**

In this paper, we have presented infinitely many Pythagorean triangles for the considered binary quadratic equation  $y^2 = 6x^2 + 1$ . As the binary quadratic equations are rich in variety, one may search for the other solutions of the considered binary quadratic equations and determine their integer solutions along with suitable properties.

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