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Negative binomial sum of random variables and modeling financial data

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Abstract
A limiting probability distribution of negative binomial sum of independently and identically distributed random variable is obtained, which we call Negative Binomial Stable (NBS) distribution. It is applied in the forecasting of market changes by fitting the pound exchange rate change in relation with the Indian rupees. Using the data we empirically compare normal, stable, geometric stable (GS) and NBS distributions. Among which the NBS distribution shows better fit than the normal, stable and GS models judged in terms of Kolmogrov distance test. The NBS model is a generalization of the GS model.

Keywords: Financial data, Geometric stable distribution, Negative binomial sum

Introduction and Preliminaries

The probability distribution of negative binomial sum of independently and identically distributed (i.i.d) random variables is studied by many authors (Cooper 1998, Ergodan *et al*. 1998) [5, 7]. Here we are presenting another application of a limiting probability distribution of negative binomial sum of i.i.d random variables, which has applications in economics, finance, management, mathematics, statistics, trade, commerce etc.

The first step towards the statistical modeling of stock price changes was taken by Bachelier (1900) [2]. His idea reflects the realities of central limit theorem: (i) independence (ii) identical distribution and (iii) finite variance of daily changes. He used the normal model, since the price change over a period of time can be regarded as the sum of changes in shorter periods of time, weekly change = sum of daily changes, daily change = sum of changes between the various transactions.

But further research in this area revealed the fact that the empirical distributions of stock returns had more kurtosis than the normal distribution (Mandelbrot 1963 a, b, and Fama 1965) [13, 14, 8]. They used symmetric stable distribution, since the family of stable distribution is the sum of i.i.d. random variables, have heavy tails and domains of attraction with respect to summation scheme. Thus the stable distributions are the only possible limits of (scaled) sum of i.i.d. random variables. This asymptotic property enables the researchers for using stable models to model this uncertainty.

Since the normal model is a special case of stable distribution, it has got some popularity among financial modelers. But a number of empirical models (Officer 1972, Blattberg *et al*. 1974, Press 1982, Akgiray *et al*. 1988) [18, 3, 20, 1] showed the inconsistency with the stable model. Because of this inconsistency DuMouchel (1973) [6] proposed a mixture of normal and stable model which has heavy tails. Boness *et al*. (1974) [4] proposed a mixture of normal distributions without heavy tails. Alternative contributions in this area of research is; Praetz (1972) [19], Blattberg *et al*. (1974) [3]. But the main demerit of all these models is, they are not based on a limiting probability distribution and do not have domains of attraction and are not closed with respect to scaling under certain transformations, that is they are not stable. The scaling idea is contributed by Mandelbrot (1977) [12].

Mitlnik and Rachev (1989, 1991, 1993) [15-17] used the stability concept of Mandelbrot and proposed Weibull distribution, which arises in the geometric summation scheme and dominate all other alternative stable models. Distributions which arise from summation scheme can be classified by the single parameter, the index of stability determining the main properties of the distribution.

Kozubowski and Rachev (1994) [10] proposed GS model and have shown that the GS model successfully compete with the normal and stable model by fitting Yen exchange rate changes (in relation to the US dollar). Where Yen is the unit of Japanese currency.
The NBS model considered in this paper belongs to the stable family, has domains of attraction and heavy tails. The model successfully competes with the normal and stable model. It also better fit the pound exchange rate change than the GS model for some specific values of the parameters. The GS model is a particular case of the NBS model.
Here is the organization of the paper: section 2 deals with preliminary definitions and properties, section 3 presents representations and simulations. Section 4 deals with application for modeling financial data. Section 5 deals with numerical simulation results and section 6 is for conclusion.

2. Preliminary Definitions and Properties

A random variable (r.v.) Y with distribution function G is said to be stable with respect to negative binominal summation scheme, if there exists a sequence of i.i.d random variables $X_1, X_2, \ldots$, a negative binomial random variable T(r,p) independent of all X_i, and constants a(p) > 0 and b(p) Є R such that

$$a(p)\sum_{i=1}^{T(r,p)}(X_i+b(p)) \xrightarrow{d} Y \text{ as } p\to 0$$

$$P(T(r,p)=k) = \binom{k+r-2}{k-1} p^r q^{k-1},\ k=1, 2, \ldots,\ 0 < p < 1,\ r > 0$$

Since T (r, p) converges in distribution to the gamma distribution with parameter r as p→ 0, Y has the characteristic function $\Psi(t) = \lambda(-\log\phi(t))$,
Where λ is the Laplace transform of the gamma distribution with parameter r and ϕ(t) is the characteristic function of an α-stable distribution.

$$\Psi(t) = (1+|t|^{\alpha}\sigma^{\alpha}\omega(t,\alpha,\beta)-i\mu t)^{-r} \tag{2.1}$$

$$\text{Where, } \omega(t,\alpha,\beta) = \begin{cases} 1-i\beta\tan(\pi\alpha/2)sign(t), \alpha\neq 1 \\ 1+i\beta(2/\pi)\ln|t|\ sign(t), \alpha\neq 1 \end{cases} \tag{2.2}$$

$$\text{and } sign(t) = \begin{cases} 1\ if\ t>0 \\ 0\ if\ t=0 \\ -1\ if\ t<0 \end{cases} \tag{2.3}$$

We call the limiting distribution Y, as $NBS_\alpha(\sigma, \beta, \mu, r)$. The parameters: α ($0 < \alpha \leq 2$) is the characteristic exponent or the index of stability, determines the main characteristics of the distribution,σ (σ > 0) is the scale parameter, β (|β| ≤) is the intensity of skewness and μ (μ ∈ R) is the shift parameter.
Proof follows from Kozubowski *et al*. (1996) [9]

Example

The gamma distribution with ch.fn. $\Psi(t) = (1-i\mu t)^{-r}$ belongs to the NBS family of distributions. The corresponding stable distribution is the degenerate distribution with ch.fn. exp(itμ).

Property 2.1

NBS distribution is closed under finite convolution for same α, σ, β, μ:

Property 2.2

Let $Y \sim NBS_\alpha(\sigma, \beta, 0, r)$ then $Yr^{\frac{-1}{\alpha}}$ is asymptotically α-stable as $r \to \infty$.

3. Estimation of Parameters: Moment Estimation

An analytical procedure which yields explicit estimators and involves minimal computation is
a version of method of moments based on sample characteristic function.Let $Y_1, Y_2, \ldots, Y_n$ be i i.d. copies of Y with characteristic function Ψ represented by (2.1)-(2.3). The function

$$\hat{\psi}_n(t) = (1/n)\sum_{j=1}^{n}(\exp(i\,t\,Y_j)$$

is called the empirical characteristic function. ofcourse

$$E(\hat{\psi}_n(t)) = \Psi(t)$$

By strong law of large numbers (SLLN)

$$\hat{\psi}_n(t) \to \Psi(t)\ (a.s)\ \text{as } n\to\infty$$

An analytic procedure which yields explicit estimator for the parameter ‘r ‘is difficult and sometimes impossible. We estimate the parameters α, σ, β and μ for different values of the parameter ‘r’
Let U(t) and V(t) be the real and imaginary parts of $\Psi(t)^{(-1/r)} - 1$ respectively.
For any $t_i > 0$ ($t_i \neq t_j$), (i,j =1,2,3,...)

$U(t_i) = |t_i|^\alpha \sigma^\alpha,\ i = 1, 2$ (3.1)

For any $t_1 \neq t_2$ ($t_1, t_2 > 0$)

$$\alpha = \frac{\log[U(t_1)/U(t_2)]}{\log\left|\frac{t_1}{t_2}\right|}$$

and for $\alpha \neq 1$

$$\log \sigma^\alpha = \log\lambda = \frac{\log|t_1|\log(U(t_2)) - \log|t_2|\log(U(t_1))}{\log\left|\frac{t_1}{t_2}\right|}$$

Replacing U(t) by its empirical counter part.

$$\hat{U}_n(t) = \frac{n^{1/r}}{\left[\left(\sum_{j=1}^{n}\cos tY_j\right)^2 + \left(\sum_{j=1}^{n}\sin tY_j\right)^2\right]^{1/2r}} \cdot \cos\left[\frac{1}{r}\operatorname{atan}\left(\frac{\sum_{j=1}^{n}\sin tY_j}{\sum_{j=1}^{n}\cos tY_j}\right)\right] - 1$$

gives the estimators $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ respectively.

$V(t) = -|t|^\alpha \sigma^\alpha \beta \tan(\pi \alpha/2) - \mu t$ (3.2)

Choose two non-zero values of t; say t_3 and t_4 such that $t_3 \neq t_4$. Then for $\alpha = 1$

$$\beta = \frac{v(t_3)t_4 - V(t_4)t_3}{t_3|t_4|^\alpha - t_3|t_4|^\alpha} \, 1/\lambda \cot\left(\frac{\pi\alpha}{2}\right)$$

$$\mu = \frac{v(t_4)|t_3|^\alpha - V(t_3)|t_4|^\alpha}{t_3|t_4|^\alpha - t_3|t_4|^\alpha}$$

Replacing V(t) by its empirical counter part

$$\hat{V}_n(t) = \frac{-n^{1/r}}{\left[\left(\sum_{j=1}^{n}\cos tY_j\right)^2 + \left(\sum_{j=1}^{n}\sin tY_j\right)^2\right]^{1/2r}} \cdot \sin\left[\frac{1}{r}\operatorname{atan}\left(\frac{\sum_{j=1}^{n}\sin tY_j}{\sum_{j=1}^{n}\cos tY_j}\right)\right]$$

gives the estimators $\hat{\beta}$ and $\hat{\mu}$ of β and μ respectively.

Case of $\alpha = 1$

Letting V(t) be the imaginary part of $\Psi(t)^{(-1/r)} - 1$.

$$\beta = \left(\frac{v(t_4)}{t_4} - \frac{v(t3)}{t_3}\right) \Big/ \left(\frac{2\sigma}{\pi}\ln\left|\frac{t_4}{t_3}\right|\right)$$

$$\mu = \left(\frac{\ln|t_3|}{t_4}V(t_4) - \frac{\ln|t_4|}{t_3}V(t_3)\right) \Big/ \left(\ln\left|\frac{t_4}{t_3}\right|\right)$$

Replacing V(t) by its empirical counter part

$$\frac{-n^{1/r}}{\left[\left(\sum_{j=1}^{n}\cos tY_j\right)^2 + \left(\sum_{j=1}^{n}\sin tY_j\right)^2\right]^{1/2r}} \cdot \sin\left[\frac{1}{r}\operatorname{atan}\left(\frac{\sum_{j=1}^{n}\sin tY_j}{\sum_{j=1}^{n}\cos tY_j}\right)\right]$$

gives the estimators $\hat{\beta}$ and $\hat{\mu}$ of β and μ respectively.

Since $\hat{\psi}_n(t) \to \Psi(t)$ (a.s), also $\hat{U}_n(t) \to U(t)$ (a.s) and $\hat{V}_n(t) \to V(t)$ (a.s), the estimators are consistent.

The procedure holds good for all choices of t ($-\infty < t < \infty$), but the rate of convergence may be different. This requires further study.

4. Representation and Simulation

The most widely used technique of simulation of random variables is the well-known inversion method (Devroye 1986). If a random variable X has the distribution funtion F, then

$$X \overset{d}{=} F^{-1}(U),$$

where F^{-1} is the inverse of F and U is a uniform random variable on (0, 1). Since generally there are no analytic expressions for the NBS distribution functions and their inverses, the inversion method is not applicable to the NBS case. The computer simulation algorithm for the NBS distribution has the following steps.

A general NBS□ (α, β μ) generator

(i) Generate a standard gamma variate Z with parameter r.
(ii) Generate a standard stable variate $X \sim S_{\square}(1, \beta, 0)$ independent of Z.
(iii) IF $\alpha \neq 1$

THEN set $Y \leftarrow \mu Z + Z^{1/\alpha}\sigma X$

ELSE set $Y \leftarrow \mu Z + Z\sigma + \sigma Z\beta\left(\frac{2}{\pi}\right)\log(Z\sigma)$

(iv) RETURN Y

Simulating a standard stable $X \sim S_{\square}(1, \beta, 0)$ random variable consists of the following steps.

1. Generate a standard exponential variate W.
2. Generate uniform $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, variate V, independent of W.
3. IF $\alpha = 1$

THEN set $X \leftarrow \frac{2}{\pi}\left[\left(\frac{\pi}{2} + \beta V\right)\tan V - \beta \log\left(\frac{W\cos V}{\frac{\pi}{2} + \beta V}\right)\right]$

ELSE

{

$$\text{Set } b \leftarrow \frac{\arctan\left(\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)}{\alpha}$$

$$\text{Set } s \leftarrow \left[1 + \beta^2 \tan^2 \frac{\pi\alpha}{2}\right]^{1/2\alpha}$$

$$\text{Set } X \leftarrow s\frac{\sin(\alpha in + b))}{(\cos V)^{1/\alpha}}\left(\frac{\cos\left(V - \alpha(V + b)\right)}{W}\right)^{(1-\alpha)/\alpha}$$

}

Following we provide a representation of the random variable Y in terms of a gamma random variable with parameter r and α - stable random variable.

Proposition 4.1

Let $Y \sim NBS_{\alpha}(\sigma, \beta, \mu, r)$ then $Y \underline{\underline{d}} \begin{cases} \mu Z + Z^{\frac{1}{\alpha}}\sigma X & ; \alpha \neq 1 \\ \mu Z + \sigma ZX + \sigma Z\beta(2/\pi)\log(Z\sigma); & \alpha = 1 \end{cases}$

where $X \sim S_{\alpha}(1, \beta, 0)$ and Z is standard gamma independent of X.
Proof follows from Kozubowski and Panorska (1996) [9]

Proposition 4.2

Let $X \sim S_{\alpha'}(\sigma, 0, 0)$ with $0 < \alpha' \leq 2$ and let $0 < \alpha < \alpha'$. Let Y be $NBS_{\alpha/\alpha'}$ random variable totally skewed to the right.
Then $W \underline{\underline{d}}\ X Y^{1/\alpha'} \sim NBS_{\alpha}(\sigma, 0, 0)$

Proof

Let $\phi_x(t)$ and $\psi_w(t)$ be the ch.fn of X and W respectively.

$\phi_x(t) = \exp(-|t|^{\alpha'}\sigma^{\alpha'})$

Since $Y \sim NBS_{\alpha/\alpha'}((\cos(\pi\alpha/2\alpha'))^{\alpha'/\alpha}, 1, 0)$

$Y \stackrel{d}{=} Z^{\alpha'/\alpha}.U$

where Z is standard gamma and $U \sim S_{\alpha/\alpha'}((\cos(\pi\alpha/2\alpha'))^{\alpha'/\alpha}\ 1, 0)$ and U has the Laplace transform $E\exp(-\gamma U) = \exp(-\gamma^{\alpha/\alpha'})$

Consider $\psi_w(t) = E\exp(it\, X\, Z^{1/\alpha}U^{1/\alpha'})$

$$= E_Z(E_U(E\exp i(tZ^{1/\alpha}U^{1/\alpha'})\,X / U) / Z),$$
$$= E_Z(E_U(\exp - |t|^{\alpha'}\sigma^{\alpha'} Z^{\alpha'/\alpha} U) / Z),$$
$$= E(\exp(-|t|^{\alpha}\sigma^{\alpha} Z)),$$
$$= (1 + |t|^{\alpha}\sigma^{\alpha})^{-r}.$$

The right side is the ch.fn. of $NBS_\alpha(\sigma,0,0)$

Which completes the proof.

Remark 4.1

Proposition 3.2 is valid for the weak limit of any random summation scheme of i.i.d random variables.

Remark 4.2

Using proposition 3.2 one can transform a symmetric NBS r.v into another symmetric NBS r.v.

Remark 4.3

By proposition 3.1 for NBS distribution moments of order less than α alone exists and if $1 < \alpha < 2$, then $E(Y) = \mu E(Z)$

5. Applications to Modeling Stocks

Here we are presenting the NBS distribution to model the pound exchange rate change in relation to Indian rupees. It is assumed that the exchange rate change in one day is the sum of a large number of changes occurred in that day. The number of changes may not be deterministic. It may be random. While using the normal and stable distributions to model the financial data, the number of changes in one day is deterministic. But it is not a realistic assumption. There may be an independent number of random innovations. Hence it is a r.v.. Kozubowski and Rechav (1994) [10] proposed the GS distribution for modeling financial data sets, where it is assumed that the number of changes in one day follows the geometric distribution and the parameter p stands for the probability of a dramatic change in the market during each period.

But in our model we are assuming that the number of changes in one day follows the negative binomial distribution. Thus if Y represents the change of a commodity in one day, thus change in one day = $\sum_{i=1}^{T(r,\rho)}$ "small changes". In terms of random variables, the daily price change $Y = X_1+X_2+ \ldots + X_{T(r,\rho)}$, where X_i represents price change between successive transactions and are assumed to be independently and identically distributed random variables. The negative binomial random variable T(r,p) represents the time at which the probabilistic structure governing the returns break down. p denotes the probability of small changes and r is time required for that change. We can observe that the price value of the shares of certain companies changes very rapidly and even we cannot record that change. These changes are occurred even within a small fraction of a second. From the table 1 we can conclude that the NBS distribution better fits those data sets where the price value changes very rapidly.

6. Numerical Simulation

We shall consider the distribution of pound exchange rate changes in relation with the Indian rupees, where pound is a unit of Britan currency. The data are daily exchange rate changes from 1-1-1990 to 31-12-1998. The data is fitted for normal, stable, geometric stable and NBS distributions. As usual we consider the change in the log (price) from t to t + 1 that is each data point $P_t = 1n(X_{t+1}) - \ln(X_t)$, where X_t represents the closing price of the commodity on day t. As shown in the histogram of data, the peaked ness is the characteristic of many financial data sets. We use maximum likelihood estimators for estimating μ and σ for the normal distribution. The parameters in stable and GS distributions are estimated by the method of moments (Kozubowski 1998, Press 1972) [21]. The technique explained in section (3) is followed for estimating parameters of NBS distribution. Observations are simulated from respective distributions and histograms are drawn for the data set and the simulated values (fig 1 - 4). Because of the scaling, the small frequencies are not visible in the histogram. Kolmogrov distance test is conducted to test the goodness of fit of the distributions.

For the normal distribution $\hat{\mu}$=0.00061943 and $\hat{\sigma}$=0.06395, for stable distribution $\hat{\alpha}$=1.5392, $\hat{\sigma}$=0.0299, $\hat{\beta}$=0.0469 and $\hat{\mu}$ =0.000594. For the normal and stable distributions the Kolmogrov distance is 0.3590 and 0.3213 respectively.The table 5.1 shows the estimated values of α, σ and μ for different values of r, of NBS distribution. The table also shows the Kolmogrov distance for simulated observations. When r = 1, we get the estimators and Kolmogrov distance of the geometric stable distribution.

Table 1: α, σ, β, μ and Kolmogrov distance for different values of r

r	α	σ	β	μ	Kolmogrov distance
0.2	1.5442	0.08617	.00127423	0.002938	0.1195
0.25	1.5432	0.07438	.001281143	0.00236	0.1367
0.50	1.5412	0.04719	.00116492	0.00120	0.1915
1	1.5402	0.0300	.001000554	0.000593	0.2377
2	1.5397	0.0191	.000871436	0.000297	0.2629
3	1.5395	0.0147	.000824166	0.000198	0.2783
4	1.5394	0.0122	.000798693	0.000149	0.2876
5	1.5394	0.0105	.000774896	0.000120	0.2913
6	1.5393	0.0093	.000781705	0.000099	0.2925
7	1.5393	0.00845	.000761809	8.48797E-05	0.2913
8	1.5390	0.00775	.000826378	7.42728E-05	0.2956
9	1.5392	0.007178	.000765571	6.60224E-05	0.2974
10	1.5392	0.006702	.000758664	5.94217E-05	0.2974
11	1.5394	0.00629	.00070948	5.40208E-05	0.2999

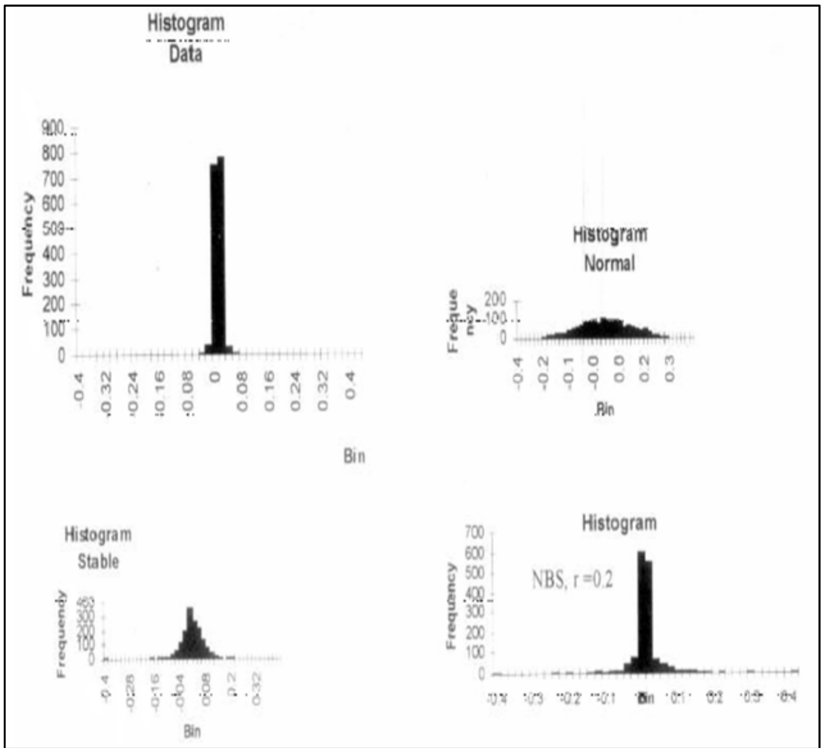

Fig 1

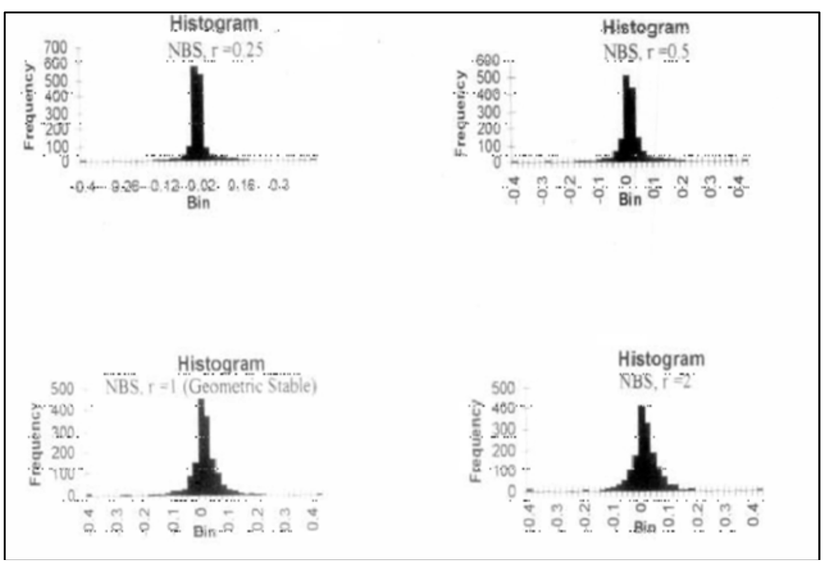

Fig 2

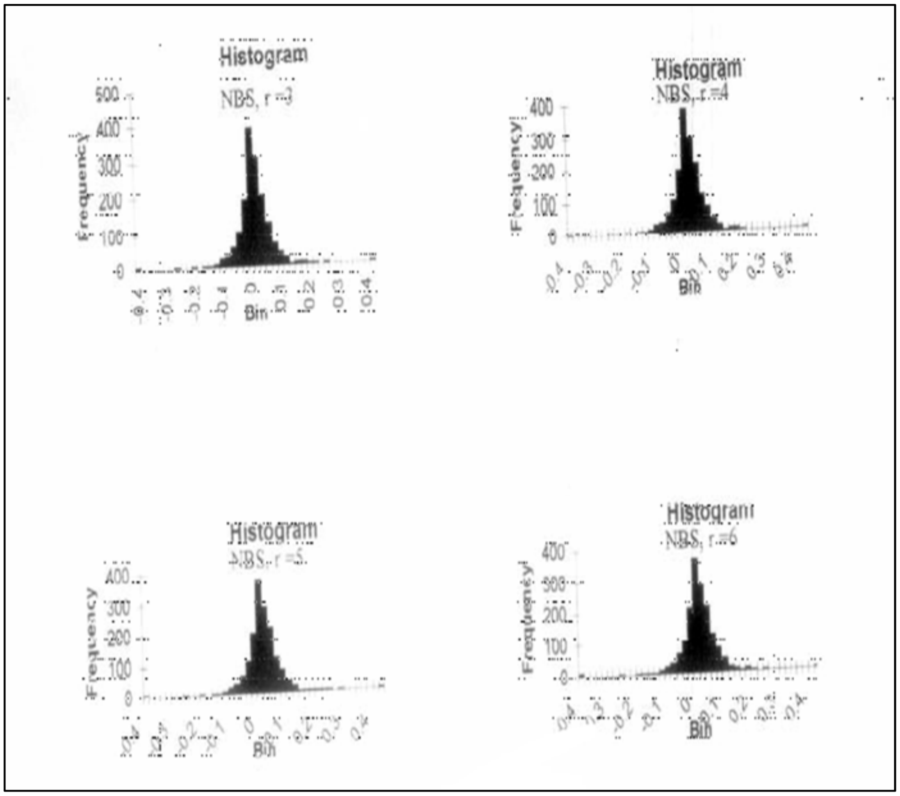

Fig 3

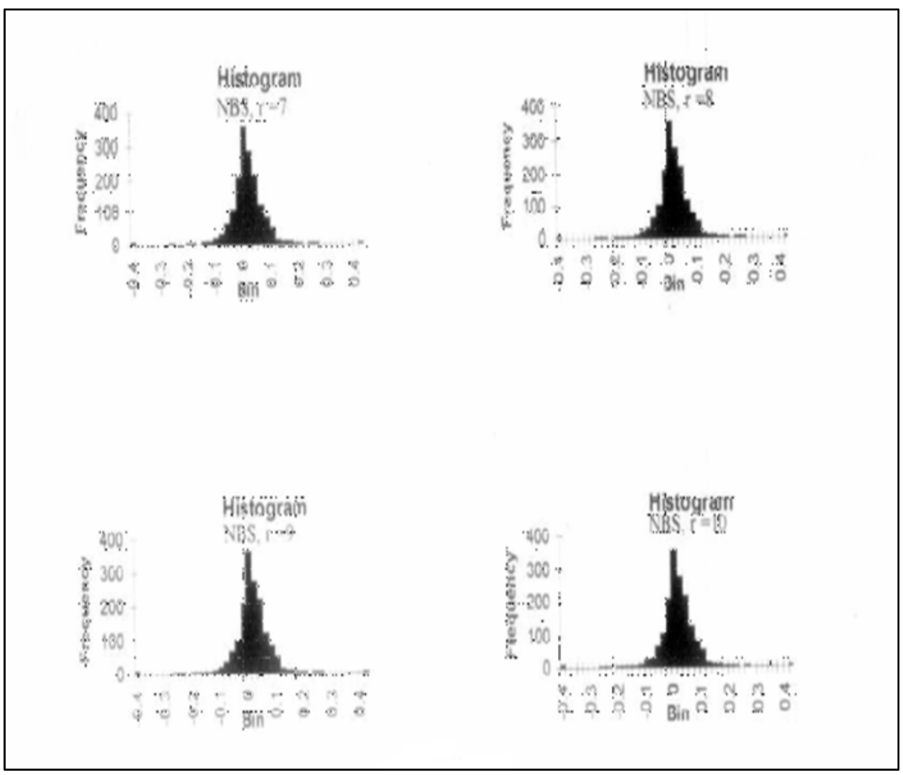

Fig 4

7. Conclusions

The Kolmogrove distance test shows that the NBS distribution better fit the pound exchange rate change than the normal and the stable model. For r = 1, we get the goodness of fit of the geometric stable distribution. For 0 < r < 1, the NBS model successfully compete with the normal, stable and GS models.

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