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New modular relations involving cubes of the Göllnitz–Gordon functions

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Abstract
Chen and Huang established some elegant modular relations for the Göllnitz–Gordon functions analogous to Ramanujan's list of forty identities for the Rogers–Ramanujan functions. In this paper, we derive some new modular relations involving cubes of the Göllnitz–Gordon functions. Furthermore, we also provide new proofs of some modular relations for the Göllnitz–Gordon functions due to Gugg.

Keywords: Modular relations; Göllnitz–Gordon functions; theta functions

1. Introduction
The aim of this paper is to present several new modular relations involving cubes of the Göllnitz–Gordon functions and provide new proofs of some known modular relations for the Göllnitz–Gordon functions due to Gugg [11]. The advantage of our method is that the identities to be proved do not need to be known in advance.
We start with an overview of the terminology and notation of q-series. Throughout this paper, we assume that $|q| < 1$ and for nonnegative integer n, we employ the standard notation

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \qquad (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$$

and

and

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

Recall that Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.1}$$

where $|ab| < 1$. By the well-known Jacobi triple product identity, the function $f(a, b)$ satisfies the following identity

$$f(a,b) = (-a, -b, ab; ab)_\infty. \tag{1.2}$$

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By (1.2), it is trivial to check that

$$f(a,b) = af(a^2b, 1/a). \tag{1.3}$$

Three special cases of (1.1) are defined by

Three special cases of (1.1) are defined by

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \quad (1.4)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}, \quad (1.5)$$

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2}. \quad (1.6)$$

For any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$f_n = (q^n; q^n)_\infty = \prod_{k=1}^{\infty} (1 - q^{nk}). \quad (1.7)$$

Replacing q by $-q$ in (1.4) and using the notation f_n, we see that

$$f(q) = \frac{f_2^3}{f_1 f_4}. \quad (1.8)$$

Replacing q by $-q$ in (1.5) and (1.6) and employing (1.8), we deduce that

$$\psi(-q) = \frac{f_1 f_4}{f_2}, \qquad \phi(-q) = \frac{f_1^2}{f_2}. \quad (1.9)$$

The well-known Rogers–Ramanujan functions are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} \quad (1.10)$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n}. \quad (1.11)$$

The functions $G(q)$ and $H(q)$ satisfy the famous Rogers–Ramanujan identities [14, 16]

$$G(q) = \frac{1}{(q, q^4; q^5)_\infty} \quad (1.12)$$

and

$$H(q) = \frac{1}{(q^2, q^3; q^5)_\infty}. \quad (1.13)$$

In a manuscript of Ramanujan [15], there are forty identities involving $G(q)$ and $H(q)$. The forty identities have been proved in a series of papers by Rogers [17], Darling [8], Watson [18], Bressoud [6], Biagioli [5] and Yesilyurt [24, 25]. Berndt *et al.* [3] published an excellent monograph on the forty identities.

The following two functions analogous to (1.12) and (1.13) are the so-called Göllnitz– Gordon functions

$$S(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2} \quad (1.14)$$

and

$$T(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n}{(q^2; q^2)_n} q^{n^2+2n}. \quad (1.15)$$

The functions $S(q)$ and $T(q)$ satisfy the following identities [9, 10]:

$$S(q) = \frac{1}{(q, q^4, q^7; q^8)_\infty} \tag{1.16}$$

and

$$T(q) = \frac{1}{(q^3, q^4, q^5; q^8)_\infty}. \tag{1.17}$$

Motivated by the similarity between the Rogers–Ramanujan functions and the Göllnitz– Gordon functions, Huang [13] and, Chen and Huang [7] derived many elegant modular relations for $S(q)$ and $T(q)$ analogous to Ramanujan's list of forty identities for the Rogers–Ramanujan functions. They extracted interesting partition theoretic results from some of the relations. Subsequently, Baruah *et al.* [1] found new proofs of modular relations which involve only $S(q)$ and $T(q)$ by using Schröter's formulas and some theta-function identities given in Ramanujan's notebooks [2]. In the process, they also found some new relations. Yan [22] also gave some proofs of modular relations due to Huang [13] and, Chen and Huang [7]. Xia and Yao [20, 23] provided new proofs of some modular relations established by Huang [13] and, Chen and Huang [7]. In the process, they also dis-covered some new relations which involve only the Göllnitz–Gordon functions. Recently, Gugg [11] discovered two new elegant relations involving cubes of the Göllnitz–Gordon functions.

We end this section by listing several modular relations involving only the Göllnitz– Gordon functions which will be proved in this paper.

Theorem 1.1: We have

$$T(q^3)S^3(q) - S(q^3)T^3(q) = 3q\frac{f^3 f_6 f^3}{f_1^2 f_3 f_4^2 f_8 f_{12}}, \tag{1.18}$$

$$S(q^3)S^3(q) + q^3 T(q^3)T^3(q) = \frac{f_6^6}{f_3^3 f_{12}^3} + 3q\frac{f_2^2 f_6^2 f_8 f_{24}}{f_1^2 f_3 f_4 f_{12}^2}, \tag{1.19}$$

$$S^3(q^3)S(q) + q^5 T^3(q^3)T(q) = \frac{f_2 f_6^3 f_8^3}{f_1 f_3^2 f_4 f_{12}^2 f_{24}}, \tag{1.20}$$

$$S^3(q^3)T(q) - q^4 T^3(q^3)S(q) = \frac{f^6}{f_1^3 f_4^3} - 3q\frac{f^2 f^2 f_8 f_{24}}{f_1 f_3^2 f_4^2 f_{12}}, \tag{1.21}$$

$$S^3(q)S(q^3) - q^3 T^3(q)T(q^3) = \frac{f_3 f_4^5}{f_1^2 f_2 f_8^2 f_{12}} + q\frac{f_2 f_8^2 f_{12}}{f_3 f_4^3}, \tag{1.22}$$

$$S^3(q)T(q^3) + T^3(q)S(q^3) = \frac{1}{q}\left(\frac{f_2 f_3 f_8^2}{f_1^2 f_4 f_{12}} - \frac{f_4^3 f_{12}}{f_2 f_3 f_8^2}\right), \tag{1.23}$$

$$S^3(q^3)S(q) - q^5 T^3(q^3)T(q) = \frac{f_4 f_{12}^3}{f_1 f_6 f_{24}^2} - q\frac{f_2 f_1 f_6 f_{24}^2}{f_3^2 f_4 f_{12}}, \tag{1.24}$$

$$S^3(q^3)T(q) + q^4 T^3(q^3)S(q) = \frac{f_1 f_{12}^5}{f_3^2 f_4 f_6 f_{24}^2} + q\frac{f_4 f_6 f_{24}^2}{f_1 f_{12}^3}, \tag{1.25}$$

$$S^3(q)S(q^5) - q^4 T^3(q)T(q^5) = \frac{f^4 f_{10}}{f_1^2 f_8^2 f_{20}} + q\frac{f^2 f^2 f_{10}}{f_1 f_4^3 f_5}, \tag{1.26}$$

$$T^3(q)S(q^5) + q S^3(q)T(q^5) = \frac{1}{q}\left(\frac{f_2^2 f_8^2 f_{10}}{f_1^2 f_4^2 f_{20}} - \frac{f_4^3 f_{10}}{f_1 f_5 f_8^2}\right), \tag{1.27}$$

$$S^3(q)S(q^9) - q^6 T^3(q)T(q^9) = \frac{f_3 f_4^3 f_{12} f_{18}}{f_1^2 f_8^2 f_9 f_{36}} + q\frac{f_2 f_6^2 f_{36}^2}{f_1 f_3 f_4^2 f_{12}}, \tag{1.28}$$

$$T^3(q)S(q^9) + q^3 S^3(q)T(q^9) = \frac{1}{q}\frac{f_2^2 f_3 f_8^2 f_{12} f_{18}}{f_1^2 f_4^3 f_9 f_{36}} - \frac{f_4 f_6^2}{f_1 f_2 f_3 f_8^2 f_{12}}. \tag{1.29}$$

Identities (1.18) and (1.20) were first discovered by Gugg [11]. Identities (1.19) and (1.21)–(1.29) are new. Moreover, we will prove the following identities:

Theorem 1.2: We have

$$\frac{S(-q^6)S(-q^3)T(q)-S(-q^6)S(q^3)T(-q)}{T(-q^6)S(q^3)T(-q)+T(-q^6)S(-q^3)T(q)} = q^5\frac{f_{16}^3 f_{24} f_{96}^2}{f_8 f_{32}^2 f_{48}^3}, \tag{1.30}$$

$$\{S(q^3)T(q)+S(-q^3)T(-q)\}\{S(q)T(q^3)-S(-q)T(-q^3)\} = 4q\frac{f_{12}^3 f_{16}^4}{f_2 f_4 f_6^3 f_8^2} \tag{1.31}$$

and

$$\{S^3(-q^3)T(q) + S^3(q^3)T(-q)\}\{T^3(-q^3)S(q) - T^3(q^3)S(-q)\} = 4q\frac{f_8^6 f_{48}^4}{f_2 f_4 f_6^4 f_{24}^4}. \tag{1.32}$$

Theorem 1.3: The following identities hold:

$$\frac{S(q^7)T(q^3) - q^2 T(q^7)S(q^3)}{S(q^{21})S(q) + q^{11}T(q^{21})T(q)} = \frac{f_1 f_{84}}{f_3 f_{28}}, \tag{1.33}$$

$$\frac{S(q^7)S(q^3) + q^5 T(q^7)T(q^3)}{S(q^{21})T(q) - q^{10}T(q^{21})S(q)} = \frac{f_4 f_{21}}{f_7 f_{12}}. \tag{1.34}$$

2. Proof of Theorem 1.1

The following 2-dissection formulas of f_1 and $\frac{1}{f_1}$ were proved by Xia and Yao [20]:

$$f_1 = f_4 S(-q^2) - q f_4 T(-q^2) \tag{2.1}$$

and

$$\frac{1}{f_1} = \frac{f_4^2}{f_2^3}S(-q^2) + q\frac{f_4^2}{f_2^3}T(-q^2). \tag{2.2}$$

Hirschhorn *et al.* [12] proved the following 2-dissection formulas for $\frac{f_3}{f_1^3}$ and $\frac{f_3^3}{f_1}$:

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7} \tag{2.3}$$

and

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.4}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}. \tag{2.4}$$

It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}
\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_6f_4^2f_{12}^2}{f_2^7} &= \frac{f_3^3}{f_1^3}=f_{12}(S(-q^6)-q^3T(-q^6))\frac{f_4^6}{f_2^9}(S(-q^2)\\
&\quad +qT(-q^2))^3 = \frac{f_4^6f_{12}}{f_2^9}(S(-q^6)-q^3T(-q^6))(S^3(-q^2)\\
&\quad +3qS^2(-q^2)T(-q^2)+3q^2S(-q^2)T^2(-q^2)+q^3T^3(-q^2))\\
&= \frac{f_4^6f_{12}}{f_2^9}(S(-q^6)S^3(-q^2)-q^6T(-q^6)T^3(-q^2)\\
&\quad +3q^2S(-q^2)T(-q^2)(S(-q^6)T(-q^2)-q^2S(-q^2)T(-q^6)))\\
&\quad +\frac{f_4^6f_{12}}{f_2^9}(q^3S(-q^6)T^3(-q^2)-q^3T(-q^6)S^3(-q^2)\\
&\quad +3qS(-q^2)T(-q^2)(S(-q^6)S(-q^2)-q^4T(-q^6)T(-q^2))).
\end{aligned} \tag{2.5}$$

Equating the even and the odd parts of (2.5), we obtain

$$\begin{aligned}
&\frac{f_4^6f_{12}}{f_2^9}(q^3S(-q^6)T^3(-q^2)-q^3T(-q^6)S^3(-q^2)+3qS(-q^2)T(-q^2)\\
&\times(S(-q^6)S(-q^2)-q^4T(-q^6)T(-q^2)))=3q\frac{f_6f_4^2f_{12}^2}{f_2^7}
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\frac{f_4^6f_{12}}{f_2^9}(S(-q^6)S^3(-q^2)-q^6T(-q^6)T^3(-q^2)+3q^2S(-q^2)T(-q^2)\\
&\times(S(-q^6)T(-q^2)-q^2S(-q^2)T(-q^6)))=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}.
\end{aligned} \tag{2.7}$$

Dividing q on both sides of (2.6) and then replacing q^2 by $-q$, in view of (1.8), we find that

$$\begin{aligned}
q(T(q^3)S^3(q)-S(q^3)T^3(q)) &= 3\frac{f_2^2f_6^4}{f_1^2f_3f_4^2f_{12}}-3S(q)T(q)(S(q^3)S(q)\\
&\quad -q^2T(q^3)T(q)).
\end{aligned} \tag{2.8}$$

By (1.16) and (1.17),

$$S(q)T(q)=\frac{f_2f_8^2}{f_1f_4^2}. \tag{2.9}$$

Gugg [11] proved that

$$S(q^3)T(q)+qS(q)T(q^3)=\frac{f_2f_4f_6^2f_{24}}{f_1f_3f_8f_{12}^2}, \tag{2.10}$$

$$S(q^3)S(q)-q^2T(q^3)T(q)=\frac{f_2^2f_6f_8f_{12}}{f_1f_3f_4^2f_{24}}. \tag{2.11}$$

It follows from (2.11), (2.8) and (2.9) that

$$S(q^3)T^3(q)-T(q^3)S^3(q)=3q^{-1}\frac{f_2^3f_6}{f_1^2f_3f_4^2f_{12}}\frac{f_8^3f_{12}^2}{f_4^2f_{24}}-\frac{f_6^3}{f_2}. \tag{2.12}$$

Replacing q by q^2 in (2.4), we have

$$\frac{f_8^3f_{12}^2}{f_4^2f_{24}} - \frac{f_6^3}{f_2} = -q^2\frac{f_{24}^3}{f_8}. \tag{2.13}$$

Identity (1.18) follows from (2.12) and (2.13).

Replacing q^2 by $-q$ in (2.7) and using (1.8), we find that

$$S(q^3)S^3(q) + q^3T(q^3)T^3(q) - 3qS(q)T(q)(S(q^3)T(q) + qS(q)T(q^3)) = \frac{f_6^6}{f_3^3f_{12}^3}. \tag{2.14}$$

Employing (2.9) and (2.10), we rewrite (2.14) in the equivalent form

$$S(q^3)S^3(q) + q^3T(q^3)T^3(q) - 3q\frac{f_2^2f_6^2f_8f_{24}}{f_1^2f_3f_4f_{12}^2} = \frac{f_6^6}{f_3^3f_{12}^3}, \tag{2.15}$$

which is nothing but (1.19).

By (2.1), (2.2) and (2.4),

$$\begin{aligned}\frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4} = \frac{f_3^3}{f_1} &= f_{12}(S(-q^6)-q^3T(-q^6))^3\frac{f_4^2}{f_2^3}(S(-q^2)+qT(-q^2))\\ &= \frac{f_4^2f_{12}^3}{f_2^3}(S^3(-q^6)-3q^3S^2(-q^6)T(-q^6)+3q^6S(-q^6)\\ &\quad T^2(-q^6)-q^9T^3(-q^6))(S(-q^2)+qT(-q^2))\\ &= \frac{f_4^2f_{12}^3}{f_2^3}(S^3(-q^6)S(-q^2)-q^{10}T^3(-q^6)T(-q^2)\\ &\quad -3q^4S(-q^6)T(-q^6)(S(-q^6)T(-q^2)-q^2T(-q^6)\\ &\quad \times S(-q^2))) + \frac{f_4^2f_{12}^3}{f_2^3}(q(S^3(-q^6)T(-q^2)-q^8T^3(-q^6)\\ &\quad \times S(-q^2)) - 3q^3S(-q^6)T(-q^6)(S(-q^6)S(-q^2)\\ &\quad - q^4T(-q^6)T(-q^2))).\end{aligned} \tag{2.16}$$

Equating the even and the odd parts on both sides of (2.16), we find that

$$\begin{aligned}\frac{f_4^3f_6^2}{f_2^2f_{12}} = \frac{f_4^2f_{12}^3}{f_2^3}&(S^3(-q^6)S(-q^2)-q^{10}T^3(-q^6)T(-q^2)-3q^4S(-q^6)\\ &\times T(-q^6)(S(-q^6)T(-q^2)-q^2T(-q^6)S(-q^2)))\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}q\frac{f_{12}^3}{f_4} = \frac{f_4^2f_{12}^3}{f_2^3}&(q(S^3(-q^6)T(-q^2)-q^8T^3(-q^6)S(-q^2))-3q^3S(-q^6)\\ &\times T(-q^6)(S(-q^6)S(-q^2)-q^4T(-q^6)T(-q^2))).\end{aligned} \tag{2.18}$$ □

Replacing q^2 by $-q$ in (2.17) and employing (1.8), we have

$$\begin{aligned}S^3(q^3)S(q) + q^5T^3(q^3)T(q) - 3q^2S(q^3)T(q^3)(S(q^3)T(q)\\ + q^3T(q^3)S(q)) = \frac{f_2^4f_6^2}{f_1f_3^2f_4f_{12}^2}.\end{aligned} \tag{2.19}$$

In view of (2.9) and (2.19), we see that

$$\begin{aligned}S^3(q^3)S(q) + q^5T^3(q^3)T(q) &= \frac{f_2^4f_6^2}{f_1f_3^2f_4f_{12}^2} + 3q^2\frac{f_2f_4f_6^3f_{24}^3}{f_1f_3^2f_8f_{12}^4}\\ &= \frac{f_2f_6^3}{f_1f_3^2f_4f_{12}^2}\quad 3q^2\frac{f_4^2f_{24}^3}{f_8f_{12}^2} + \frac{f_6^3}{f_2}.\end{aligned} \tag{2.20}$$

Replacing q by $-q$ in (2.3) and using (1.8), we obtain

$$\frac{f_2^9}{f_1^3 f_4^3 f_6} = \frac{f_4^3}{f_{12}} + 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{2.21}$$

Identity (1.20) follows from (2.20) and (2.21).

Dividing q on both sides of (2.18), then replacing q^2 by $-q$ and employing (2.9), we find that

$$\frac{f_2^6}{f_1^3 f_4^3} = S^3(q^3)T(q) - q^4T^3(q^3)S(q) + 3q\,S(q^3)T(q^3)(S(q^3)S(q) - q^2T(q^3)T(q)). \tag{2.22}$$

It follows from (2.11), (2.9) and (2.22) that (1.21) is true.

Xia and Yao [21] proved the following 2-dissection formulas for f_1^2 and f_1f_3:

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8} \tag{2.23}$$

and

$$f_1f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{2.24}$$

See also Xia [19]. In view of (2.1), (2.23) and (2.24),

$$\begin{aligned}
&\frac{f_2^2 f_8^7 f_{12}^4}{f_4^4 f_6 f_{16}^2 f_{24}^2} + 2q^2\frac{f_4^4 f_6 f_{16}^2 f_{24}^2}{f_2 f_8^3 f_{12}^2} - q\left(2\frac{f_2^2 f_8 f_{12}^4 f_{16}^2}{f_4^2 f_6 f_{24}^2} + \frac{f_4^2 f_6 f_8^3 f_{24}^2}{f_{12}^2 f_{16}^2}\right)\\
&= \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right)\left(\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}\right) = f_1^3 f_3\\
&= f_4^3 f_{12}(S(-q^2) - qT(-q^2))^3(S(-q^6) - q^3T(-q^6))\\
&= f_4^3 f_{12}(S^3(-q^2)S(-q^6) + 3q^2S(-q^2)T^2(-q^2)S(-q^6)\\
&\quad + 3q^4S^2(-q^2)T(-q^2)T(-q^6) + q^6T^3(-q^2)T(-q^6))\\
&\quad + f_4^3 f_{12}(-q^3S^3(-q^2)T(-q^6) - 3q^5S(-q^2)T^2(-q^2)T(-q^6) -\\
&\quad 3q\,S^2(-q^2)T(-q^2)S(-q^6) - q^3T^3(-q^2)S(-q^6)).
\end{aligned} \tag{2.25}$$

Equating the even parts and the odd parts on both sides of (2.25), we have

$$\begin{aligned}
&S^3(-q^2)S(-q^6) + 3q^2S(-q^2)T^2(-q^2)S(-q^6)\\
&\quad + 3q^4S^2(-q^2)T(-q^2)T(-q^6) + q^6T^3(-q^2)T(-q^6)\\
&\quad = \frac{f_2^2 f_8^7 f_{12}^3}{f_4^7 f_6 f_{16}^2 f_{24}^2} + 2q^2\frac{f_4 f_6 f_{16}^2 f_{24}^2}{f_2 f_8^3 f_{12}^3}
\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}
&3S^2(-q^2)\,T(-q^2)\,S(-q^6) + q^2T^3(-q^2)\,S(-q^6) + q^2S^3(-q^2)\,T(-q^6)\\
&\quad + 3q^4\,S(-q^2)T^2(-q^2)T(-q^6) = 2\frac{f_2^2 f_8 f_{12}^3 f_{16}^2}{f_4^5 f_6 f_{24}^2} + \frac{f_6 f_8^3 f_{24}^2}{f_4 f_{12}^3 f_{16}^2}.
\end{aligned} \tag{2.27}$$

Huang [13] discovered the following two identities:

$$S(q^3)S(q) + q^2T(q^3)T(q) = \frac{f_3 f_4}{f_1 f_{12}} \tag{2.28}$$

and

$$S(q^3)T(q) - qS(q)T(q^3) = \frac{f_1 f_{12}}{f_3 f_4}. \qquad (2.29)$$

Baruah *et al.* [1] and Xia and Yao [20, 23] gave new proofs of (2.28) and (2.29). Replacing ^{2} by $-q$ in (2.26) and employing (1.8), (2.9) and (2.29), we obtain

$$S^3(q)S(q^3) - q^3T^3(q)T(q^3)$$

$$= \frac{f_3 f_4^5}{f_1^2 f_2 f_8^2 f_{12}} - 2q\frac{f_2 f_8^2 f_{12}}{f_3 f_4^3} + 3qS(q)T(q)(S(q^3)T(q) - qS(q)T(q^3))$$

$$= \frac{f_3 f_4^5}{f_1^2 f_2 f_8^2 f_{12}} + q\frac{f_2 f_8^2 f_{12}}{f_3 f_4^3}, \qquad (2.30)$$

which is nothing but (1.22).

Replacing q^2 by $-q$ in (2.27), we see that

$$-q(S^3(q)T(q^3) + T^3(q)S(q^3)) = 2\frac{f_2 f_3 f_8^2}{f_1^2 f_4 f_{12}} + \frac{f_4^3 f_{12}}{f_2 f_3 f_8^2}$$

$$- 3S(q)T(q)(S(q^3)S(q) + q^2T(q)T(q^3)), \qquad (2.31)$$

which implies (1.23) by employing (2.9) and (2.28).

It follows from (2.1), (2.23) and (2.24) that

$$\frac{f_2 f_8^2 f_{12}^2 f_{24}^3}{f_4^2 f_{48}^2} + 2q^4\frac{f_4^4 f_6^2 f_{24} f_{48}^2}{f_2 f_8^2 f_{12}^2} - q\frac{f_4^4 f_6^2 f_{24}^7}{f_2 f_8^2 f_{12}^4 f_{48}^2} + 2q^2\frac{f_2 f_8^2 f_{12}^4 f_{48}^2}{f_4^2 f_{24}^3}$$

$$= \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2} \quad \frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}} = f_1 f_3^3$$

$$= f_4 f_{12}^3(S(-q^2) - qT(-q^2))(S(-q^6) - q^3T(-q^6))^3$$

$$= f_4 f_{12}^3(S(-q^2)S^3(-q^6) + 3q^4S^2(-q^6)T(-q^6)T(-q^2)$$

$$+ 3q^6S(-q^6)T^2(-q^6)S(-q^2) + q^{10}T^3(-q^6)T(-q^2))$$

$$- qf_4 f_{12}^3(S^3(-q^6)T(-q^2) + 3q^2S^2(-q^6)T(-q^6)S(-q^2)$$

$$+ 3q^6S(-q^6)T^2(-q^6)T(-q^2) + q^8S(-q^2)T^3(-q^6)). \qquad (2.32)$$

Equating the even parts and the odd parts on both sides of (2.32), we see that

$$S(-q^2)S^3(-q^6) + q^{10}T^3(-q^6)T(-q^2) = \frac{f_2 f_8^2 f_{24}^3}{f_4^3 f_{12} f_{48}^2} + 2q^4\frac{f_4^3 f_6^2 f_{24} f_{48}^2}{f_2 f_8^2 f_{12}^5}$$

$$-3q^4S(-q^6)T(-q^6)(S(-q^6)T(-q^2) + q^2T(-q^6)S(-q^2)) \qquad (2.33)$$

and

$$S^3(-q^6)T(-q^2) + q^8T^3(-q^6)S(-q^2) = \frac{f_4^3 f_6^2 f_{24}^7}{f_2 f_8^2 f_{12}^7 f_{48}^2} + 2q^2\frac{f_2 f_8^2 f_{12} f_{48}^2}{f_4^3 f_{24}^3}$$

$$- 3q^2S(-q^6)T(-q^6)(S(-q^6)S(-q^2) + q^4T(-q^6)T(-q^2)). \qquad (2.34)$$

Replacing q^2 by $-q$ in (2.33) and utilizing (1.8), (2.9) and (2.29), we can obtain (1.24). In view of (1.8), (2.9), (2.28) and (2.34), we can derive (1.25) after replacing q^2 by $-q$ in (2.34).

Xia and Yao [20] proved that

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.35}$$

Thanks to (2.1), (2.2), (2.23) and (2.35),

$$\begin{aligned}
&\frac{f_2^2 f_8^6 f_{20}^3}{f_4^3 f_{10}^3 f_{16}^2 f_{40}} - q\frac{f_2 f_8^4 f_{40}}{f_{10}^2 f_{16}^2} - 2q\frac{f_2^2 f_{16}^2 f_{20}^3}{f_4 f_{10}^3 f_{40}} + 2q^2\frac{f_2 f_4^2 f_{16}^2 f_{40}}{f_8^2 f_{10}^2}\\
&= \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right)\left(\frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}\right) = \frac{f_1^3}{f_5}\\
&= \frac{f_4^3 f_{20}^2}{f_{10}^3}(S(-q^2) - qT(-q^2))^3(S(-q^{10}) + q^5T(-q^{10}))\\
&= \frac{f_4^3 f_{20}^2}{f_{10}^3}\big(S^3(-q^2)S(-q^{10}) + 3q^2S(-q^2)T^2(-q^2)S(-q^{10})\\
&\quad -3q^6S^2(-q^2)T(-q^2)T(-q^{10}) - q^8T^3(-q^2)T(-q^{10})\big)\\
&\quad + q\frac{f_4^3 f_{20}^2}{f_{10}^3}\big(-3S^2(-q^2)T(-q^2)S(-q^{10}) - q^2T^3(-q^2)S(-q^{10})\\
&\quad + q^4S^3(-q^2)T(-q^{10}) + 3q^6S(-q^2)T^2(-q^2)T(-q^{10})\big).
\end{aligned} \tag{2.36}$$

Extracting the even parts and the odd parts in (2.36), we deduce that

$$\begin{aligned}
\frac{f_2^2 f_8^6 f_{20}}{f_4^6 f_{16}^2 f_{40}} + 2q^2\frac{f_2 f_{10} f_{16}^2 f_{40}}{f_4 f_8^2 f_{20}^2} &= S^3(-q^2)S(-q^{10}) + 3q^2S(-q^2)\\
&\quad\times T^2(-q^2)S(-q^{10}) - 3q^6S^2(-q^2)T(-q^2)T(-q^{10})\\
&\quad - q^8T^3(-q^2)T(-q^{10})
\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
-\frac{f_2 f_8^4 f_{10} f_{40}}{f_4^3 f_{16}^2 f_{20}^2} - 2\frac{f_2^2 f_{16}^2 f_{20}}{f_4^4 f_{40}} &= -3S^2(-q^2)T(-q^2)S(-q^{10})\\
&\quad - q^2T^3(-q^2)S(-q^{10}) + q^4S^3(-q^2)T(-q^{10})\\
&\quad + 3q^6S(-q^2)T^2(-q^2)T(-q^{10}).
\end{aligned} \tag{2.38}$$

Replacing q^2 by $-q$ in (2.37) and (2.38) and using (1.8), we get

$$\begin{aligned}
\frac{f_4^4 f_{10}}{f_1^2 f_8^2 f_{20}} - 2q\frac{f_2^2 f_8^2 f_{10}}{f_1 f_4^3 f_5} &= S^3(q)S(q^5) - q^4T^3(q)T(q^5)\\
&\quad - 3qS(q)T(q)(T(q)S(q^5) - q^2S(q)T(q^5))
\end{aligned} \tag{2.39}$$

and

$$\begin{aligned}
-\frac{f_4^3 f_{10}}{f_1 f_5 f_8^2} - 2\frac{f_2^2 f_8^2 f_{10}}{f_1^2 f_4^2 f_{20}} &= qT^3(q)S(q^5) + q^2S^3(q)T(q^5) - 3S(q)T(q)\\
&\quad\times(S(q)S(q^5) + q^3T(q)T(q^5)).
\end{aligned} \tag{2.40}$$

Huang [13] established the following two identities:

$$S(q^5)S(q) + q^3T(q^5)T(q) = \frac{f_2 f_{10}}{f_1 f_{20}} \tag{2.41}$$

and

$$S(q^5)T(q) - q^2T(q^5)S(q) = \frac{f_2 f_{10}}{f_4 f_5}; \qquad (2.42)$$

See also [1, 20, 23]. Identity (1.26) follows from (2.9), (2.39) and (2.42). Combining (2.9), (2.40) and (2.41), we arrive at (1.27). Xia and Yao [20] also established the following 2-dissection formula for

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}. \qquad (2.43)$$

Replacing q by $-q$ in (2.43) and employing (1.8), we get

$$\frac{f_1}{f_9} = \frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q\frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}. \qquad (2.44)$$

In view of (2.1), (2.2), (2.23) and (2.44),

$$\frac{f_2^2 f_8^5 f_{12}^3}{f_4^3 f_6 f_{16}^2 f_{18}^2} - q\frac{f_2 f_6 f_8^5 f_{36}^2}{f_4 f_{12} f_{16}^2 f_{18}^3} - 2q\frac{f_2^2 f_{12}^3 f_{16}^2}{f_4 f_6 f_8 f_{18}^2} + 2q^2\frac{f_2 f_4 f_6 f_{16}^2 f_{36}^2}{f_8 f_{12} f_{18}^3}$$

$$= \left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}\right)\left(\frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q\frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}\right) = \frac{f_1^3}{f_9}$$

$$= \frac{f_4^3 f_{36}^2}{f_{18}^3}\left(S(-q^2) - qT(-q^2)\right)^3\left(S(-q^{18}) + q^9T(-q^{18})\right)$$

$$= \frac{f_4^3 f_{36}^2}{f_{18}^3}\big(S^3(-q^2)S(-q^{18}) - 3qS^2(-q^2)T(-q^2)S(-q^{18}) + 3q^2S(-q^2)T^2(-q^2)S(-q^{18}) - q^3T^3(-q^2)S(-q^{18})$$

$$+q^9S^3(-q^2)T(-q^{18}) - 3q^{10}S^2(-q^2)T(-q^2)T(-q^{18})$$
$$+ 3q^{11}S(-q^2)T^2(-q^2)T(-q^{18}) - q^{12}T^3(-q^2)T(-q^{18})\big). \qquad (2.45)$$

Extracting the even parts and the odd parts in (2.45), we deduce that

$$\frac{f_2^2 f_8^5 f_{12}^3 f_{18}}{f_4^6 f_6 f_{16}^2 f_{36}^2} + 2q^2\frac{f_2 f_6 f_{36}^2}{f_4^2 f_8 f_{12}} = S^3(-q^2)S(-q^{18}) + 3q^2S(-q^2)T^2(-q^2)$$
$$\times S(-q^{18}) - 3q^{10}S^2(-q^2)T(-q^2)$$
$$\times T(-q^{18}) - q^{12}T^3(-q^2)T(-q^{18}) \qquad (2.46)$$

and

$$-\frac{f_2 f_6 f_8^5}{f_4^4 f_{12} f_{16}^2} - 2\frac{f_2^2 f_{12}^3 f_{16}^2 f_{18}}{f_4^4 f_6 f_8 f_{36}^2} = -3S^2(-q^2)T(-q^2)S(-q^{18})$$
$$- q^2T^3(-q^2)S(-q^{18}) + q^8S^3(-q^2)T(-q^{18}) + 3q^{10}S(-q^2)$$
$$\times T^2(-q^2)T(-q^{18}). \qquad (2.47)$$

Replacing q^2 by $-q$ in (2.46) and (2.47) and using (1.8), we obtain

$$\frac{f_3f_4^3f_{12}f_{18}}{f_1^2f_8^2f_9f_{36}} - 2q\frac{f_2f_6^2f_8^2}{f_1f_3f_4^2f_{12}} = S^3(q)S(q^9) - q^6T^3(q)T(q^9)$$
$$- 3qS(q)T(q)(S(q^9)T(q) - q^4S(q)T(q^9)) \tag{2.48}$$

and

$$\frac{f_4^4f_6^2}{f_1f_2f_3f_8^2f_{12}} + 2\frac{f_2^2f_3f_8^2f_{12}f_{18}}{f_1^2f_4^3f_9f_{36}} = -qT^3(q)S(q^9) - q^4S^3(q)T(q^9)$$
$$+ 3S(q)T(q)(S(q)S(q^9) + q^5T(q)T(q^9)). \tag{2.49}$$

Huang [13] proved the following two identities:

$$S(q^9)S(q) + q^5T(q^9)T(q) = \frac{f_2f_3f_{12}f_{18}}{f_1f_4f_9f_{36}} \tag{2.50}$$

and

$$S(q^9)T(q) - q^4T(q^9)S(q) = \frac{f_6^2}{f_3f_{12}}; \tag{2.51}$$

Baruah *et al.* [1] and, Xia and Yao [20, 23] also gave proofs of (2.50) and (2.51). Identity (1.28) follows from (2.9), (2.48) and (2.51) and identity (1.29) follows from (2.9), (2.49) and (2.50). This completes the proof.

3. Proof of Theorem 1.2

In view of (1.2), (1.16) and (1.17), it is easy to check that

$$S(q^3)T(-q) = \frac{1}{(q^3, q^{12}, q^{21}; q^{24})_\infty(-q^3, q^4, -q^5; q^8)_\infty}$$
$$= \frac{f_8f(-q^5, -q^{19})f(-q^{11}, -q^{13})}{f_4f_{12}f_{24}(q^6, q^{10}; q^{16})_\infty}. \tag{3.1}$$

It is trivial to obtain the following identity from Entry 29 on page 45 of [2]:

$$f(a, b)f(c, d) = f(ac, bd)f(ad, bc) + af(b/c, ac^2d)f(b/d, acd^2), \tag{3.2}$$

where $ab = cd$. Setting $a = -q^5$, $b = -q^{19}$, $c = -q^{11}$ and $d = -q^{13}$ in (3.2), we find that

$$f(-q^5, -q^{19})f(-q^{11}, -q^{13}) = f(q^{16}, q^{32})f(q^{18}, q^{30})$$
$$- q^5f(q^8, q^{40})f(q^6, q^{42}). \tag{3.3}$$

It follows from (1.2), (3.1) and (3.3) that

$$S(q^3)T(-q) = \frac{f_8f_{32}f_{48}^3(-q^{18}, -q^{30}; q^{48})_\infty}{f_4f_{12}f_{16}f_{24}f_{96}(q^6, q^{10}; q^{16})_\infty}$$
$$- q^5\frac{f_{16}^2f_{96}(-q^6, -q^{42}; q^{48})_\infty}{f_4f_{12}f_{32}(q^6, q^{10}; q^{16})_\infty}, \tag{3.4}$$

which yields

$$S(q^3)T(-q) + S(-q^3)T(q) = 2\frac{f_8 f_{32} f_{48}^3(-q^{18}, -q^{30}; q^{48})_\infty}{f_4 f_{12} f_{16} f_{24} f_{96}(q^6, q^{10}; q^{16})_\infty}$$

and

$$S(-q^3)T(q) - S(q^3)T(-q) = 2q^5\frac{f_{16}^2 f_{96}(-q^6, -q^{42}; q^{48})_\infty}{f_4 f_{12} f_{32}(q^6, q^{10}; q^{16})_\infty}.$$

By (1.16), (1.17), (3.5) and (3.6),

$$\frac{S(-q^3)T(q) - S(q^3)T(-q)}{S(q^3)T(-q) + S(-q^3)T(q)} = q^5\frac{f_{16}^3 f_{24} f_{96}^2 T(-q^6)}{f_8 f_{32}^2 f_{48}^3 S(-q^6)}.$$

Identity (1.30) follows from (3.7).

By (1.2), (1.16) and (1.17), it is a routine to verify that

$$\begin{aligned} S(q^3)T(q) &= \frac{f_8 f_{24}}{f_4 f_{12}(q^3, q^3, q^5, q^{11}, q^{13}, q^{19}, q^{21}, q^{21}; q^{24})_\infty} \\ &= \frac{f_8 f(q^3, q^{21}) f(q^3, q^{21}) f(q^5, q^{19}) f(q^{13}, q^{11})}{f_4 f_{12} f_{24}^3 (q^6, q^{10}; q^{16})_\infty (q^6, q^{42}; q^{48})_\infty} \\ &= q^{13}\frac{f_8 f(q^3, q^{21}) f(q^{21}, q^3) f(q^{19}, q^5) f(q^{37}, q^{-13})}{f_4 f_{12} f_{24}^3 (q^6, q^{10}; q^{16})_\infty (q^6, q^{42}; q^{48})_\infty}. \end{aligned}$$

where $ab = cd$. Substituting $a = q^3$, $b = q^{21}$, $c = q^{19}$, $d = q^5$ and $n = q^{18}$ in (3.9), we see that

$$\begin{aligned} &f(q^3, q^{21}) f(q^{21}, q^3) f(q^{19}, q^5) f(q^{37}, q^{-13}) \\ &\quad - f(-q^3, -q^{21}) f(-q^{21}, -q^3) f(-q^{19}, -q^5) f(-q^{37}, -q^{-13}) \\ &\quad = 2q^3 f(q^8, q^{16}) f(q^{-16}, q^{40}) f(q^6, q^{18}) \psi(q^{24}). \end{aligned} \quad (3.5)$$

By (1.2), (1.3) and (1.5), it is trivial to check that

$$2q^3 f(q^8, q^{16}) f(q^{-16}, q^{40}) f(q^6, q^{18}) \psi(q^{24}) = 2q^{-13}\frac{f_{48}^2}{f_{24}} f^2(q^8, q^{16}) \times f(q^6, q^{18}). \quad (3.6)$$

In view of (1.3), (3.8), (3.10) and (3.11),

$$S(q^3)T(q) + S(-q^3)T(-q) = 2\frac{f_{12} f_{16}^2}{f_4 f_6 f_8 (q^6, q^{10}; q^{16})_\infty (q^6, q^{42}; q^{48})_\infty}. \quad (3.7)$$

Also, by (1.2), (1.16) and (1.17),

$$\begin{aligned} S(q)T(q^3) &= \frac{f_8 f_{24}}{f_4 f_{12}(q, q^7, q^9, q^9, q^{15}, q^{15}, q^{17}, q^{23}; q^{24})_\infty} \\ &= \frac{f_8 f(q, q^{23}) f(q^7, q^{17}) f(q^9, q^{15}) f(q^{15}, q^9)}{f_4 f_{12} f_{24}^3 (q^2, q^{14}; q^{16})_\infty (q^{18}, q^{30}; q^{48})_\infty}. \end{aligned} \quad (3.8)$$

Setting $a = q$, $b = q^{23}$, $c = q^9$, $d = q^{15}$ and $n = q^6$ in (3.9), we obtain

$$f(q, q^{23}) f(q^7, q^{17}) f(q^9, q^{15}) f(q^{15}, q^9) - f(-q, -q^{23}) f(-q^7, -q^{17}) \times f(-q^9, -q^{15}) f(-q^{15}, -q^9) = 2q f^2(q^8, q^{16}) f(q^6, q^{18}) \psi(q^{24}). \quad (3.14)$$

It follows from that (1.2), (1.5), (3.13) and (3.14) that

$$S(q)T(q^3) - S(-q)T(-q^3) = 2q \frac{f_{12} f_{16}^2}{f_4 f_6 f_8 (q^2, q^{14}; q^{16})_\infty (q^{18}, q^{30}; q^{48})_\infty}. \quad (3.9)$$

Identity (1.31) follows from (3.12) and (3.15).
Thanks to (1.11), (1.16) and (1.17),

$$\begin{aligned} S^3(-q^3)T(q) &= \frac{f_8 f_{24}^3}{f_4 f_{12}^3 (-q^3, -q^{21}; q^{24})_\infty^3 (q^3, q^5, q^{11}, q^{13}, q^{19}, q^{21}; q^{24})_\infty} \\ &= \frac{f_8 f_{24}^3 (q^3, q^3, -q^5, -q^{11}, -q^{13}, -q^{19}, q^{21}, q^{21}; q^{24})_\infty}{f_4 f_{12}^3 (q^6, q^{42}; q^{48})_\infty^3 (q^{10}, q^{22}, q^{26}, q^{38}; q^{48})_\infty} \\ &= \frac{f_8 f(-q^3, -q^{21}) f(-q^3, -q^{21}) f(q^5, q^{19}) f(q^{11}, q^{13})}{f_4 f_{12}^3 f_{24} (q^6, q^{42}; q^{48})_\infty^2 (q^6, q^{10}; q^{16})_\infty} \\ &= \frac{q^5 f_8 f(-q^3, -q^{21}) f(-q^3, -q^{21}) f(q^{-5}, q^{29}) f(q^{11}, q^{13})}{f_4 f_{12}^3 f_{24} (q^6, q^{42}; q^{48})_\infty^2 (q^6, q^{10}; q^{16})_\infty}. \end{aligned} \quad (3.10)$$

Setting $a = -q^3$, $b = -q^{21}$, $c = q^{-5}$, $d = q^{29}$ and $n = -q^8$ in (3.9), we see that

$$\begin{aligned} &f(-q^3, -q^{21}) f(-q^3, -q^{21}) f(q^{-5}, q^{29}) f(q^{11}, q^{13}) \\ &\quad - f(q^3, q^{21}) f(q^3, q^{21}) f(-q^{-5}, -q^{29}) f(-q^{11}, -q^{13}) \\ &= -2q^3 f(-q^{-8}, -q^{32}) f(q^6, q^{18}) f(-q^8, -q^{16}) \psi(q^{24}) \\ &= 2q^{-5} f(q^6, q^{18}) f^2(-q^8, -q^{16}) \psi(q^{24}). \end{aligned} \quad (3.11)$$

Thanks to (1.2), (1.5), (3.16) and (3.17),

$$S^3(-q^3)T(q) + S^3(q^3)T(-q) = 2 \frac{f_8 f(q^6, q^{18}) f^2(-q^8, -q^{16}) \psi(q^{24})}{f_4 f_{12}^3 f_{24} (q^6, q^{42}; q^{48})_\infty^2 (q^6, q^{10}; q^{16})_\infty}. \quad (3.12)$$

Also, in view of (1.2), (1.3), (1.16) and (1.17),

$$\begin{aligned} T^3(-q^3)S(q) &= \frac{f_8 f_{24}^3}{f_4 f_{12}^3 (-q^9, -q^{15}; q^{24})_\infty^3 (q, q^7, q^9, q^{15}, q^{17}, q^{23}; q^{24})_\infty} \\ &= \frac{f_8 f_{24}^3 (q^9, q^9, q^{15}, q^{15}; q^{24})_\infty (-q, -q^7, -q^{17}, -q^{23}; q^{24})_\infty}{f_4 f_{12}^3 (q^{18}, q^{30}; q^{48})_\infty^2 (q^2, q^{14}; q^{16})_\infty} \\ &= \frac{f_8 f(q, q^{23}) f(q^7, q^{17}) f(-q^9, -q^{15}) f(-q^{15}, -q^9)}{f_4 f_{12}^3 f_{24} (q^{18}, q^{30}; q^{48})_\infty^2 (q^2, q^{14}; q^{16})_\infty}. \end{aligned} \quad (3.13)$$

Taking $a = q$, $b = q^{23}$, $c = q^7$, $d = q^{17}$ and $n = -q^8$ in (3.9), we find that

$$\begin{aligned} &f(q, q^{23}) f(q^7, q^{17}) f(-q^9, -q^{15}) f(-q^{15}, -q^9) \\ &\quad - f(-q, -q^{23}) f(-q^7, -q^{17}) f(q^9, q^{15}) f(q^{15}, q^9) \\ &= 2q f(q^6, q^{18}) f^2(-q^8, -q^{16}) \psi(q^{24}). \end{aligned} \quad (3.14)$$

By (3.19) and (3.20),

$$T^3(-q^3)S(q) - T^3(q^3)S(-q) = 2q \frac{f_8 f(q^6, q^{18}) f^2(-q^8, -q^{16}) \psi(q^{24})}{f_4 f_{12}^3 f_{24} (q^{18}, q^{30}; q^{48})_\infty^2 (q^2, q^{14}; q^{16})_\infty}. \quad (3.21)$$

4. Proof of Theorem 1.3

Besides the functions $\psi(q)$, $\phi(q)$ and $f(-q)$, Ramanujan defined one further function $\chi(q) := (-q; q^2)_\infty$. (4.1) By the definition of f_n, it is trivial to verify that

$$\chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad \chi(-q) = \frac{f_1}{f_2} \tag{4.2}$$

Employing (2.1), (2.2) and (4.2), we see that

$$\begin{aligned}\chi(q^3)\chi(-q^7) &= \frac{f_6^2 f_7}{f_3 f_{12} f_{14}} = \frac{f_6^2}{f_{12} f_{14}} f_{28}(S(-q^{14}) - q^7 T(-q^{14})) \\ &\quad \times \frac{f_{12}^2}{f_6^3}(S(-q^6) + q^3 T(-q^6)) \\ &= \frac{f_{12} f_{28}}{f_6 f_{14}}(S(-q^{14})S(-q^6) - q^{10} T(-q^{14})T(-q^6)) \\ &\quad + q^3 \frac{f_{12} f_{28}}{f_6 f_{14}}(S(-q^{14})T(-q^6) - q^4 T(-q^{14})S(-q^6)), \end{aligned} \tag{4.3}$$

which yields

$$\begin{aligned}\chi(q^3)\chi(-q^7) - \chi(-q^3)\chi(q^7) &= 2q^3 \frac{f_{12} f_{28}}{f_6 f_{14}}(S(-q^{14})T(-q^6) \\ &\quad - q^4 T(-q^{14})S(-q^6)) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\chi(q^3)\chi(-q^7) + \chi(-q^3)\chi(q^7) &= 2\frac{f_{12} f_{28}}{f_6 f_{14}}(S(-q^{14})S(-q^6) \\ &\quad - q^{10} T(-q^{14})T(-q^6)). \end{aligned} \tag{4.5}$$

Similarly, it follows from (2.1), (2.2) and (4.2) that

$$\begin{aligned}\chi(q)\chi(-q^{21}) &= \frac{f_2^2 f_{21}}{f_1 f_4 f_{42}} = \frac{f_2^2}{f_4 f_{42}} f_{84}(S(-q^{42}) - q^{21} T(-q^{42})) \\ &\quad \times \frac{f_4^2}{f_2^3}(S(-q^2) + q T(-q^2)) \\ &= \frac{f_4 f_{84}}{f_2 f_{42}}(S(-q^{42})S(-q^2) - q^{22} T(-q^{42})T(-q^2)) \\ &\quad + q \frac{f_4 f_{84}}{f_2 f_{42}}(S(-q^{42})T(-q^2) - q^{20} T(-q^{42})S(-q^2)), \end{aligned} \tag{4.6}$$

which implies that

$$\begin{aligned}\chi(q)\chi(-q^{21}) + \chi(-q)\chi(q^{21}) &= 2\frac{f_4 f_{84}}{f_2 f_{42}}(S(-q^{42})S(-q^2) \\ &\quad - q^{22} T(-q^{42})T(-q^2)) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\chi(q)\chi(-q^{21}) - \chi(-q)\chi(q^{21}) &= 2q\frac{f_4 f_{84}}{f_2 f_{42}}(S(-q^{42})T(-q^2) \\ &\quad - q^{20} T(-q^{42})S(-q^2)). \end{aligned} \tag{4.8}$$

Recently, Berndt and Yesilyurt [4] proved the following identity:

$$\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}=q^3\,\frac{\phi(-q^4)\psi(-q^{42})}{\phi(-q^{12})\psi(-q^{14})}. \tag{4.9}$$

In view of (1.9), (4.4), (4.7) and (4.9), we have

$$\frac{S(-q^{14})T(-q^6)-q^4T(-q^{14})S(-q^6)}{S(-q^{42})S(-q^2)-q^{22}T(-q^{42})T(-q^2)}=\frac{f_4^3f_6f_{24}f_{168}}{f_2f_8f_{12}^3f_{56}}, \tag{4.10}$$

Recently, Berndt and Yesilyurt [4] proved the following identity:

$$\frac{\chi(q^3)\chi(-q^7)-\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})+\chi(-q)\chi(q^{21})}=q^3\,\frac{\phi(-q^4)\psi(-q^{42})}{\phi(-q^{12})\psi(-q^{14})}. \tag{4.9}$$

In view of (1.9), (4.4), (4.7) and (4.9), we have

$$\frac{S(-q^{14})T(-q^6)-q^4T(-q^{14})S(-q^6)}{S(-q^{42})S(-q^2)-q^{22}T(-q^{42})T(-q^2)}=\frac{f_4^3f_6f_{24}f_{168}}{f_2f_8f_{12}^3f_{56}}, \tag{4.10}$$

which yields (1.33) after replacing q^2 by $-q$ and using (1.8).

Berndt and Yesilyurt [4] also established the following elegant identity:

$$q\,\frac{\chi(q^3)\chi(-q^7)+\chi(-q^3)\chi(q^7)}{\chi(q)\chi(-q^{21})-\chi(-q)\chi(q^{21})}=\frac{\phi(-q^{84})\psi(-q^2)}{\phi(-q^{28})\psi(-q^6)}. \tag{4.11}$$

It follows from (1.9), (4.5), (4.8) and (4.11) that

$$\frac{S(-q^{14})S(-q^6)-q^{10}T(-q^{14})T(-q^6)}{S(-q^{42})T(-q^2)-q^{20}T(-q^{42})S(-q^2)}=\frac{f_8f_{14}f_{56}f_{84}^3}{f_{24}f_{28}^3f_{42}f_{168}}. \tag{4.12}$$

Replacing q^2 by $-q$ in (4.12) and employing (1.8), we can easily arrive at (1.34). This completes the proof.

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