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Continued finite fractions

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Abstract
Representation of a real number x as finite continued fraction of the form.

$$x = \alpha_0 + \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \cfrac{\beta_3}{\alpha_3 + \cfrac{-}{- + \cfrac{\beta_n}{\alpha_n}}}}}$$

Where $\alpha_0, \alpha_1, \alpha_2, \ldots \ldots \alpha_n$ *and* $\beta_1, \beta_2, \ldots \ldots \ldots \beta_n$ are integers. In this result we define convergent of a finite continued fraction with positive quotients and discuss fraction algorithm and Euclid's algorithm.

Keywords: Euclid Algorithm, Fraction, Real and Fraction

1. Introduction
Define a function

$$f(n) = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{-}{\cfrac{-}{- + \cfrac{1}{\alpha_N}}}}}} \qquad \ldots\ldots\ldots\ldots\ldots\ (a)$$

Consisting of N+1 variables $\alpha, \alpha_1, \alpha_2, \ldots \ldots \alpha_N$ as a finite continued fraction. As the representation (a) is difficult, we shall usually write it as $[\alpha_0, \alpha_1, \alpha_2, \ldots \ldots \alpha_N]$ and we call $\alpha_0, \alpha_1, \alpha_2, \ldots \ldots \alpha_N$ the partial quotients or simply quotients of the finite continued fraction. As above we see that

$[\alpha_0] = \frac{\alpha_0}{1}, [\alpha_0, \alpha_1] = \frac{\alpha_0\alpha_1+1}{\alpha_1}, [\alpha_0, \alpha_1, \alpha_2] = \frac{\alpha_2\alpha_1\alpha_0+\alpha_2+\alpha}{\alpha_2\alpha_1+1}$,…….Therefore $[\alpha_0, \alpha_1] = \alpha_0 + \frac{1}{\alpha_1}$ and similarly $[\alpha_0, \alpha_1, \ldots \ldots \ldots . \alpha_{n-1}, \alpha_n] = [\alpha_0, \alpha, \ldots \ldots \ldots \ldots \alpha_{n-2,} \alpha_{n-1} + \frac{1}{\alpha_n}]$ …(1.1)

i.e. $[\alpha_0, \alpha_1, \ldots \ldots \ldots \alpha_n] = \alpha_0 + \frac{1}{[\alpha_0, \alpha_1, \ldots\ldots, \alpha_n]} = [\alpha_0[\alpha_0, \alpha, \ldots \ldots \alpha_n]]$, for $1 \leq n \leq$N

Moreover

$[\alpha_0, \alpha_1 \ldots \ldots \ldots \alpha_n] = [\alpha_0, \alpha, \ldots \ldots \ldots \alpha_{m-1}, [\alpha_m, \alpha_{m+1}, \ldots \ldots \ldots ., \alpha]]$ for $1 \leq n \leq N$

1.1 Definition: the quantity $[\alpha_0, \alpha, \dots\dots\dots \alpha_n]$ for $(1 \le n \le N)$ is called nth convergent to$[\alpha_0, \alpha_1, \dots\dots\dots \alpha_N]$. Also gt is easy to find the convergent by means of the following theorem.

Theorem 1.2 let p_n and q_n be defined as under $p_0 = \alpha, p_1 = \alpha_1\alpha_0 + 1, p_n = \alpha_n p_{n-1} + p_{n-2}$ where $(2 \le n \le N)$ and $q_1 = 1, q_1 = \alpha_1\ q_n = \alpha_n\ q_{n-1} + q_{n-2}\ (2 \le n \le N)\ then\ [\alpha_0, \alpha, \dots\dots\dots \alpha_n] = \frac{p_n}{q_n}$.

<u>Proof</u>: - for n=1 theorem is obviously true.
Let suppose that result hold for $n \le m$ where $m < N$ then $[\alpha_0, \alpha, \dots\dots\dots.. \alpha_{m-1}, \alpha_m]$
$= \frac{p_m}{q_m} = \frac{\alpha_m p_{m-1} + p_{m-2}}{\alpha_m q_{m-1} + q_{m-2}}$ and $p_{m-1}, p_{m-2}, q_{m-1}, q_{m-2}\ depend\ only\ upon\ \alpha_0, \alpha_1, \dots\dots\dots \alpha_{m-1}$.
Hence using (1.1) we get $[\alpha_0, \alpha_1, \dots\dots. \alpha_{m-1}, \alpha_m, \alpha_{m+1}]$
$= [\alpha_0, \alpha, \dots\dots\dots.. \alpha_{m-1}, \alpha_m + \frac{1}{\alpha_{m+1}}]$
$= \frac{\left(\alpha_m + \frac{1}{\alpha_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(\alpha_m + \frac{1}{\alpha_{m+1}}\right) q_{m-1} + q_{m-2}}$
$= \frac{\alpha_{m+1}(\alpha_m p_{m-1} + p_{m-2}) + p_{m-1}}{\alpha_{m+1}(\alpha_m q_{m-1} + q_{m-2}) + q_{m-1}}$
$= \frac{\alpha_{m+1} p_m + p_{m-1}}{\alpha_{m+1} q_m + q_{m-1}} = \frac{p_{m+1}}{q_{m+1}}$
Hence by induction, the theorem is proved.

Note 1:- From $p_0 = \alpha_0, p_1 = \alpha_1\alpha_0 + 1, p_n = \alpha_n p_{n-1} + p_{n-2} (2 \le n \le N)$ and

$$q_1 = 1, q_1 = \alpha_1, \dots\dots\dots. q_n = \alpha_n\ q_{n-1} + q_{n-2} (2 \le n \le N)$$

It follows that $\frac{p_n}{q_n} = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$
Also $p_n q_{n-1} - p_{n-1} q_n = (\alpha_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(\alpha_n q_{n-1} + q_{n-2})$
$= (p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$
Repeating the argument with n-1, n-2, ………,2 in the place of n we get

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}(p_1 q_0 - p_0 q_1) = (-1)^{n-1}$$

Also $p_n q_{n-2} - p_{n-2} q_n = (\alpha_n\ p_{n-1} + p_{n-2})\ q_{n-2} - p_{n-2}(\alpha_n q_{n-1} + q_{n-2)}$
$= \alpha_n(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^{n-1}\alpha_n$.

Note 2:- The functions $p_n$and q_n satisfies the following.

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}\ or\ \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$$

Also they satisfy $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1}\alpha_n\ or\ \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1}\alpha_n}{q_{n-2} q_n}$
1.3 Definition:- Now we assign numerical values to the quotients α_n so to the fraction

$$\alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\dots\dots\dots\dots\dots + \cfrac{1}{\alpha_N}}}}}$$

and to its convergent.
Now suppose that $\alpha_1 > 0, \alpha_2 > 0, \dots\dots\dots \alpha_N > 0$. α_0 may be negative in this case the continued fraction is said to be simple. Write $x_n = \frac{p_n}{q_n}, x = x_n$ so that the value of the continued fraction is $x_n\ or\ x$. Then

$$[\alpha_0, \alpha_1, \dots\dots.. \alpha_N] = [\alpha_0, \alpha, \dots\dots\dots \alpha_{n-1}, [\alpha_n, \alpha_{n+1}, \dots\dots\dots \alpha_N]] = \frac{[\alpha_n, \alpha_{n+1}, \dots\dots., \alpha_n]\ p_{n-1} + p_{n-2}}{[\alpha_n, \alpha_{n+1}, \dots\dots., \alpha_n] q_{n-1} + q_{n-2}}$$

For $2 \le n \le N$
Note: - as every q_n is positive then from $\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1}\alpha_n}{q_{n-2} q_n}$ and $\alpha_1 > 0, \alpha_2 > 0, \dots\dots\dots \alpha_N > 0$, $x_n - x_{n-2}\ has\ the\ sign\ of\ (-1)^n$.
Which proves that the even convergent x_{2n} increase strictly with n, while the odd convergent x_{2n+1} decrease strictly.
Also from $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$, $x_n - x_{n-1}$ has the sign of $(-1)^{n-1}$
So that $x_{2m+1} > x_{2m}$ of possible we suppose that $x_{2m+1} \le x_{2\mu}\ for\ some\ m, \mu. If\ m < \mu$ then from above $x_{2m+1} < x_{2m}\ and\ m < \mu\ then\ x_{2\mu+1} < x_{2\mu}\ which\ is\ a\ contradiction.$
Therefore we say that every odd convergent is greater than any even convergent.

1.4 Definition:- If all α_n are integers then the continued fraction is called Simple Fraction. If p_n and q_n are integers and q_n is positive then $[\alpha_0, \alpha_1, \dots \dots . . \alpha_N]=\frac{p_N}{q_N}$ =x we say that the number x (which is necessarily rational) is represented by continued fraction.

Theorem 1.5: $q_n \geq n$, with inequality when n>3

Proof: in the first place, $q_0 = 1, q_1 = \alpha_1 \geq 1. If\ n \geq 2$ then $q_n = \alpha_n q_{n-1} + q_{n-2} \geq q_{n-1} + 1$ so that $q_n > q_{n-1}$ and $q_n \geq n$. If n>3 then

$$q_n \geq q_{n-1} + q_{n-2} > q_{n-1} + 1 \geq n, and\ so\ q_n \geq n.$$

1.6 Definition:- Any simple continued fraction $[\alpha_0, \alpha_1, \dots \dots . \alpha_N]$ represents a rational number $x = x_N$

Theorem 1.7: If x is representable by a simple continued fraction with an odd (even) number of convergent, it is also representable by one with an even (odd) number.

Proof:- If $\alpha_n \geq 2\ then\ [\alpha_0, \alpha_1, \dots \dots \dots \alpha_n] = [\alpha_0, \alpha_1, \dots \dots \alpha_{n-1,}\ \alpha_{n-1}, 1]$

of $\alpha_n = 1,$

For example [2,2,3]=[2,2,2,1] this choice of alternative representations is often useful. We call $\alpha'_n, = [\alpha_n, \alpha_{n+1}, \dots \dots \alpha_N]$ $(0 \leq n \leq N)$ the nth complete quotient of the continued fraction $[\alpha_0, \alpha_1, \dots \dots \dots . . \alpha_N]$. Thus $x = \alpha'_0, x = \frac{\alpha'_1 \alpha_0 + 1}{\alpha'_1}\ and$

$x = \frac{\alpha'_n p_{n-1} + p_{n-2}}{\alpha'_n q_{n-1} + q_{n-2}}$ $(2 \leq n \leq N)$…….. (b)

Theorem 1.8: of two simple continued fractions $[\alpha_0, \alpha_1, \dots \dots \dots, \alpha_n]$ and$[\beta_0, \beta_1, \dots \dots . ., \beta_n]$ have the same value x, and$[\beta_m > 1$ then M=N and the fractions are identical.

Proof:- when we say that the two continued fractions are identical we mean that they are formed by the same sequence of partial quotients.

We have $\alpha_0 = [x] = \beta_0$. Let us suppose that the first n partial quotients in the continued fraction are identical and that α'_n and β'_n are the nth complete quotients. Then $x = [\alpha_0, \alpha_1, \dots \dots . \alpha_{n-1}, \alpha'_n] = [\alpha_0, \alpha_1, \dots \dots \dots \alpha_{n-1}, b'_n]$

Of n=1 then $\alpha_0 + \frac{1}{\alpha'_1} = \ \alpha_0 + \frac{1}{\beta'_1}, \alpha'_1 = \beta'_1$

Therefore $\alpha_1 = \beta_1$

Of $n > 1$ then by $\frac{\alpha'_n \beta_{n-1} + p_{n-2}}{\alpha'_n q_{n-1} + q_{n-2}} = \frac{\beta'_n p_{n-1} + p_{n-2}}{\beta'_n q_{n-1} + q_{n-2}}$

$$(\alpha'_n - b'_n)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = 0$$

But $p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^n$

Then $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} and\ so\ \alpha_n = \beta_n$

Suppose now for example, that $N \leq M$ then all argument show that $\alpha_n = \beta_n$ for $N \leq M$ of $M > N$ then $\frac{p_N}{q_N}$

$=[\alpha_0, \alpha_1, \dots \dots \dots \alpha_n]$=$[\alpha_0, \alpha_1, \dots \dots \dots \alpha_n], \beta_{N+1}, \dots \dots \dots . ., \beta_M]$

$=\frac{\beta'_{n+1} p_n + p_{n-1}}{\beta'_{n+1} q_n + q_{n-1}} hence\ p_n q_{n-1} - p_{n-1} q_n = 0$

Which is false hence m=n and fractions are identical.

2. Continued Fraction Algorithm and Euclid's Algorithm

Let x be any real number, and let $x_0 = [x]$

Then $x = \alpha_O + \xi_0, 0 \leq \xi_0 < 1$

If $\xi_0 \neq 0\ we\ write\ \frac{1}{\xi_0} = \alpha'_1, [\alpha'_n] = \alpha_1, \alpha'_1 = \alpha_1 + \xi_1. 0 \leq \xi, < 1.$

If $\xi_1 \neq 0\ we\ can\ write\ \frac{1}{\xi_1} = \alpha'_2 = \alpha_2 + \xi_2; 0 \leq \xi_2 < 1\ and\ so\ on$

Also $\alpha'_n = \frac{1}{\xi_{n-1}} > 1\ and\ so\ \alpha_n \geq 1, for\ n \geq 1$

Thus $x = [\alpha_0, \alpha'_1] = \left[\alpha_0, \alpha_1 + \frac{1}{\xi'_2}\right] = [\alpha_0, \alpha_1, \alpha'_2] = [\alpha_0, \alpha_1, \alpha_2, \alpha'_3] \dots \dots ..$

where $\alpha_0, \alpha_1, \alpha_2, \dots \dots ..$ are integers and $\alpha_1 > 0, \alpha_2 > 0, \dots ..$

the system of equation $x = \alpha_0 + \xi_0 (0 \leq \xi_0 < 1),$

$\frac{1}{\xi_0} = \alpha'_1 = \alpha + \xi_1, (0 \leq \xi_1 \leq 1),$

$\frac{1}{\xi_1} = \alpha'_2 = \alpha_2 + \xi_2, (0 \leq \xi_2 \leq 1),$

…………is known as the continued fraction algorithm. The algorithm continuous so long as $\xi_n \neq 0$. if we eventually reach a value of n, say N, for which $\xi_N = 0$, the algorithm terminates and $x = [\alpha_0, \alpha_1, \dots \dots . \alpha_N]$

In this case x is repeated by a simple continued fraction and is rational. The number α'_n are the complete quotients of the continued fraction.

Theorem 2.1 Any rational number can be represented by a finite simple continued fraction.

Proof:- if x is an integer, then $\xi_0 = 0\ and\ x = \alpha_0. If\ x\ is\ not\ integer, then\ x = \frac{h}{k}, where\ h\ and\ k\ are\ integers\ and\ k > 1.$

Since $\frac{h}{k} = \alpha_0 + \xi_0, h = \alpha_0 k + \xi_0 k, \alpha_0 is\ the\ quotient\ and\ k_1 = \xi_0 k$ the remainder when h is divided by k_1.

If $\xi_0 \neq 0\ then\ \alpha'_1 = \frac{1}{\xi_0} = \frac{k}{k_1} = \alpha_1 + \xi_1,$

$$k = \alpha_1 k_1 + \xi_1 k_1 : thus\ \alpha_1 is\ th\ quotient, and\ k_2 = \xi_1 k_1 the\ remainder, when\ k\ is\ divided\ by\ k_1.$$

Thus we obtain a series of equations $h = \alpha_0 k + k_1$,
$k = \alpha_1 k_1 + k_2, k_1 = \alpha_2 k_2 + k_3, \ldots \ldots \ldots$ Continuing so long $\xi_n \neq 0$ or what is the same thing so long as $k_{n+1} \neq 0$
The non-negative integers$k, k_1, k_2,$.....form a strictly decreasing sequence and so $k_{n+1} = 0$ $for\ some\ N$. It follows that $\xi_N = 0$ for some N and the continued fraction algorithm terminates. This proves the theorem.

Remark:- the system of equations

$$h = \alpha_0 k + k, (0 < k_1 < k),$$
$$k = \alpha_1 k_1 + k_2, (0 < k_2 < k_1)$$

……………………………

…………………………....

$$k_{N-2} = \alpha_{N-1} k_{N-1} + k_N, (0 < k_N, < k_{n-1})$$
$$k_{N-1} = \alpha_N k_N \text{ is known as euclid's algoithm}$$

3. Difference between the Fraction and It's Convergent

Suppose $N > 1$ and $n > 0$ $then$ by $x = \frac{\alpha'_n p_{n-1} + p_{n-2}}{\alpha'_n q_{n-1} + q_{n-2}}, (1 \leq n \leq N-1)$

$$x - \frac{p_n}{q_n} = -\frac{p_n q_{n-1} - p_{n-1} q_n}{q_n(\alpha'_{n+1} q_n + q_{n-1})} = \frac{(-1)^n}{q_n(\alpha'_{n+1} q_n + q_{n-1})}$$

Also $x - \frac{p_0}{q_0} = x - \alpha_0 = \frac{1}{\alpha'_1}$

If we write $q'_1 = \alpha'_1, q^1_n = \alpha'_n q_{n-1} + q_{n-2}, (1 \leq n \leq N-1)$

Theorem 3.1:- If $(1 \leq n \leq N-1)$ then $x - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q'_{n+1}}$

Proof:- $\alpha_{n+1} < \alpha'_{n+1} < \alpha_{n+1} + 1$ for $n \leq N-2$,

We have $\alpha_n < \alpha'_n < \alpha_n + 1$ $(0 \leq n \leq N-1$, except that $\alpha'_{N-1} = \alpha_{N-1} + 1$

When $\alpha_N = 1$ hence of we neglect this case for the moment, we have $q_1 = \alpha_1 < \alpha'_1 + 1 \leq q_2$ and $q'_{n+1} = \alpha'_{n+1} q_n + q_{n-1} > \alpha_{n+1} q_n + q_{n-1} = q_{n+1}$

$$q'_{n+1} < \alpha_{n+1} q_n + q_{n-1} + q_n = q_{n+1} + q_n \leq \alpha_{n+2} q_{n+1} + q_n = q_{n+2}$$

For $1 \leq n \leq N-2$, it follows that $\frac{1}{q_{n+2}} <$ ।$p_n - q_n x$ । $< \frac{1}{q_{n+1}} (n \leq N-2)$, while ।$p_{n-1} - q_{n-1} x$ । $= \frac{1}{q_N}, p_N - q_N x = 0.$ $in\ the\ exceptional\ case$ $q'_{n+1} < \alpha_{n+1} q_n + q_{n-1} + q_n = q_{n+1} + q_n \leq \alpha_{n+2} q_{n+1} q_n = q_{n+2}$ $must\ be\ replaced\ by$ $q'_{n-1} = (\alpha_{N-1} + 1)\ q_{N-2} + q_{N-3} = q_{N-1} + q_{N-2} = \alpha_N$ and the first inequality. In the case 1 $\frac{1}{q_{n+2}} <$ ।$p_n - q_n x$ । $< (n \leq N-2)$ by an equality. In this case show that । $p_n - q_n x$ । decreases as n increase. Since

$$q_n \text{ increase, } \left| x - \frac{p_n}{q_n} \right| \text{ decrease}$$

We sum up the most important conclusion in the following theorem

i.e. of N>1, n>0 then the differences $x - \frac{p_n}{q_n}, q_n x - p_n = \frac{(-1)^n \delta n}{q_{n+1}}$

where $0 < \delta_n < 1$ $(1 \leq n \leq N-2)$,

$\delta_{n-1} = 1$, $\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$ for $n \leq N-1$ with inequality in both places except when n=N-1.

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