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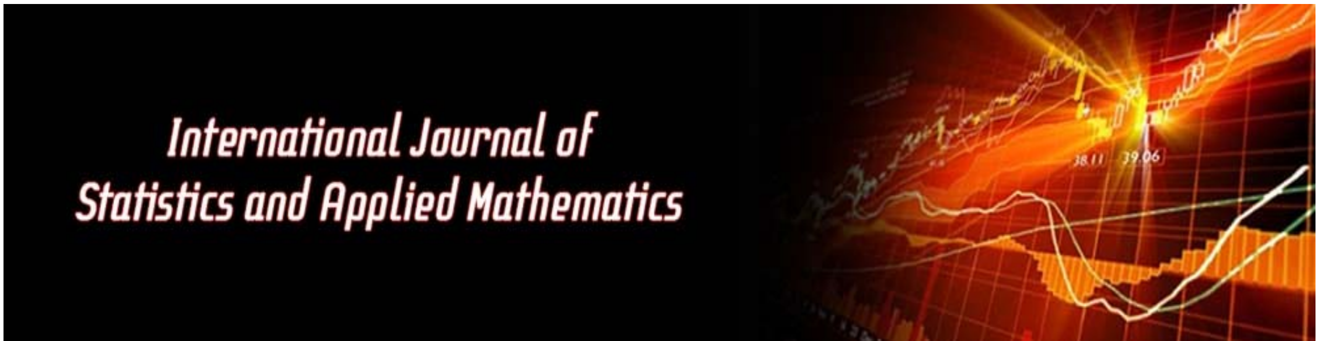

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Significance of C* algebra in f functional analysis

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Abstract
C*-algebras are an area of research in functional analysis, a branch of mathematics. A C*-algebra is a complex algebra A of continuous linear operators on a complex Hilbert space with two additional properties:

- A is a topologically closed set in the norm topology of operators.
- A is closed under the operation of taking adjoints of operators.

Keywords: algebra, adjoint, C* identity

Introduction
C*-algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933.
Subsequently, John von Neumann attempted to establish a general framework for these algebras which culminated in a series of papers on rings of operators. These papers considered a special class of C*-algebras which are now known as von Neumann algebras.
Around 1943, the work of Israel Gelfand and Mark Naimark yielded an abstract characterization of C*-algebras making no reference to operators on a Hilbert space.
C*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics. Another active area of research is the program to obtain classification, or to determine the extent of which classification is possible, for separable simple nuclear C*-algebras.

C*-Algebra
We begin with the abstract characterization of C*-algebras given in the 1943 paper by Gelfand and Naimark.
A C*-algebra, *A*, is a Banach algebra over the field of complex numbers, together with a map $*: A \to A$. One writes x^* for the image of an element x of A. The map * has the following property:

- It is an involution, for every x in A

$$x^{**} = (x^*)^* = x$$

- For all x, y in A:

$$(x+y)^* = x^* + y^*$$
$$(xy)^* = y^* x^*$$

- For every complex number λ in **C** and every x in A:

$$(\lambda x)^* = \overline{\lambda} x^*.$$

- For all x in A:

$$\|x^* x\| = \|x\| \|x^*\|.$$

Let L = (H, B, z) be an LP-type system. The number

$$\delta := \max \{|B| \,|\, B \in B\}$$

is called the combinatorial dimension of L.

Let B ⊆ G ⊆ H, B a basis. An element j E G is called enforced in (G, B) if z(B) > z(G – {j}): The number

$$\delta(G,B) := \min(\delta,|G|) - \neq \{ j \in G \mid j\, enforced\ in (G,B) \}$$

is called the hidden dimension of (G,B).
Note that if j is enforced in (G,B), then j E B,

$$\delta\ (G,B) \geq \min(\delta,|G|) - |B| \geq 0.$$

then z(B) = z(G) and B is a basis of G, as one might have expected. To see this, consider any other basis B' ⊆ G. Since $|B| = \min(\delta,|G|)$ holds, which implies |B'| ≤ |B|, there exists an element j E B – B', so B' ⊆ G – {j}, which in turn implies z(B') ≤ z(G – {j}) < z(B). Thus B is optimal and even unique with this property among all bases contained in G which proves the claim.
As we show next, two ingredients make algorithm RF-L Ptype efficient, and this is where the randomization comes in. First, if the hidden dimension is small compared to |G| (which in particular is the case if ff is small compared to |G|), the second recursive call in line 10 is necessary only with small probability. Second, in compensation for the unlucky event that the second call does become necessary, the hidden dimension decreases quickly on average.
We start with analyzing the unlucky event. To this end consider the elements {$j_{1,\ldots\ldots\ldots}$, j_k} that actually cause a second recursive call if they are chosen in line 5, together with their respective bases of line 6 and 9,

$$B_l^{'} = B\left(G - \{j_l\}\right)$$
$$B_l^{''} = \uparrow \left(B_l^{'}, jl\right) \qquad , l = 1,\ldots\ldots,k.$$

Assume the j_l are ordered by increasing z-values of the B_l, such that

$$z\left(G - \{j_1\}\right) \leq \ldots \leq z\left(G - \{j_k\}\right).$$

Theorem: If *A* is a C*-sub-algebra of *K*(*H*), then there exists Hilbert spaces $\{H_i\}_{i\in I}$ such that

$$A \cong \bigoplus_{i\in I} K(H_i),$$

where the (C*-)direct sum consists of elements (T_i) of the Cartesian product Π $K(H_i)$ with $\|T_i\| \to 0$.
Though *K*(*H*) does not have an identity element, a sequential approximate identity for *K*(*H*) can be developed. To be specific, *H* is isomorphic to the space of square summable sequences l^2; we may assume that $H = l^2$. For each natural number *n* let H_n be the subspace of sequences of l^2 which vanish for indices $k \leq n$ and let e_n be the orthogonal projection onto H_n. The sequence $\{e_n\}_n$ is an approximate identity for *K*(*H*).
K(*H*) is a two-sided closed ideal of *B*(*H*). For separable Hilbert spaces, it is the unique ideal. The quotient of *B*(*H*) by *K*(*H*) is the Calkin algebra.
Let *X* be a locally compact Hausdorff space. The space $C_0(X)$ of complex-valued continuous functions on *X* that *vanish at infinity* (defined in the article on local compactness) form a commutative C*-algebra $C_0(X)$ under point-wise multiplication and addition. The involution is point-wise conjugation. $C_0(X)$ has a multiplicative unit element if and only if X is compact. As does any C*-algebra, $C_0(X)$ has an approximate identity.
In the case of $C_0(X)$ this is immediate: consider the directed set of compact subsets of X, and for each compact K let f_K be a function of compact support which is identically 1 on K. Such functions exist by the Tietze extension theorem which applies to locally compact Hausdorff spaces. Function sequence $\{f_K\}$ is an approximate identity.
The Gelfand representation states that every commutative C*-algebra is *-isomorphic to the algebra $C_0(X)$, where X is the space of characters equipped with the weak* topology. Furthermore if $C_0(X)$ is isomorphic to $C_0(Y)$ as C*-algebras, it follows that X and Y are homeomorphic. This characterization is one of the motivations for the non-commutative topology and non-commutative geometry programs.

Significance of C* Algebra
The algebra M(*n*, **C**) of *n* × *n* matrices over **C** becomes a C*-algebra if we consider matrices as operators on the Euclidean space, $\mathbf{C}^n$, and use the operator norm ||.|| on matrices. The involution is given by the conjugate transpose. More generally, one can consider finite direct sums of matrix algebras. In fact, all C*-algebras that are finite dimensional as vector spaces are of this form, up to isomorphism. The self-adjoint requirement means finite-dimensional C*-algebras are semisimple, from which fact one can deduce the following theorem of Artin–Wedderburn type:
$\left(G, B_l^{''}\right)$ has hidden dimension at most $\delta\left(G,B\right) - l$.

Proof. Because of $z\left(B_l^{''}\right) > z\left(B\right)$ all elements which are enforced in (G,B) are also enforced in $\left(G, B_l^{''}\right)$. Moreover,

$$z\left(B_l^{''}\right) > z\left(B_l^{'}\right) = z\left(G - \{j_l\}\right) \geq z\left(G - \{j_{l-1}\}\right) \geq \ldots \geq z\left(G - \{j_1\}\right),$$

So $\left(G, B_l^{''}\right)$ features at least ` new enforced elements $j_{1,\ldots\ldots\ldots}$, j_L.
Since the hidden dimension cannot get negative, this also shows that no more than $\left(G, B_l^{''}\right)$ elements j E G - B can trigger a second recursive call.

Corollary
Consider the call RF-LPtype(G,B), B != G, and let B' be the basis of G-{j} computed in line 6 of the algorithm. Then

$$\text{prob}\left(z\left(B' \cup \{J\}\right) > z\left(B'\right)\right) \leq \frac{\delta\ (G,B)}{|G - B|},$$

where the probability is over all choices of j E G – B.

Theorem. A finite-dimensional C*-algebra, *A*, is canonically isomorphic to a finite direct sum

$$A = \bigoplus_{e \in \min A} Ae$$

where min *A* is the set of minimal nonzero self-adjoint central projections of *A*.

Each C*-algebra, *Ae*, is isomorphic (in a non-canonical way) to the full matrix algebra M(dim(*e*), **C**). The finite family indexed on min *A* given by $\{\dim(e)\}_e$ is called the *dimension vector* of *A*. This vector uniquely determines the isomorphism class of a finite-dimensional C*-algebra. In the language of K-theory, this vector is the positive cone of the K_0 group of *A*.

An immediate generalization of finite dimensional C*-algebras are the approximately finite dimensional C*-algebras.

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