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The general Liapounov theorem for some degenerate differential systems

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Abstract
The aim of this article is to generalize the famous Liapounov Equation: $VT + T^*V = -2S$ related to the stationary system : $x'(t) = Tx(t)$, to the general equation $A^*WB + B^*WA = -2S$, where S , V and W are uniformly positive operators, using the spectral theory of the pencil of operators $\lambda A - B$ corresponding to the degenerate differential system :

$$Ax'(t) = Bx(t),\ t \geq 0,$$

where A and B are bounded operators in a Hilbert space.

Keywords: Liapounov Equation, Degenerate Differential System, Spectral Theory, Pencil of operators

1. Introduction

In Control Theory we often use systems (Continuous or Discrete) in the form:

$$x'(t) = Tx(t) + F(t, x(t))\ ;\ t \geq 0, \text{ or} \tag{1}$$

$$x_{n+1} = Tx_n + F_n(x_n)\ ;\ n = 0,1,2,..., \tag{2}$$

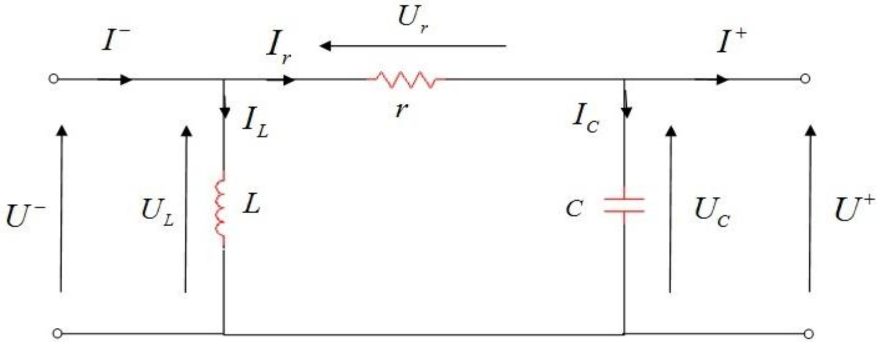
where T is a linear operator, or a matrix in finite-dimensional spaces. Since 1970 many mathematicians as Favini A, Yagi A, Campbell SL, Mahwin J, Rutkas AG [3,4], Vlasenko LA [5]. are interested in the more general implicit or degenerate systems of the form:

$$Ax'(t) = Bx(t) + F(t, x(t))\ ;\ t \geq 0, \text{ or} \tag{3}$$

$$Ax_{n+1} = Bx_n + F_n(x_n)\ ;\ n=0,1,2,..., \tag{4}$$

where A and B are two linear operators. Furthermore, A is not necessarily invertible.

Example: Consider the following Electrical Circuit as shown in the following figure [3]



Fig

where I^- is the current intensity, U^- is the voltage difference, r is the resistance, L is the induction, and C is the capacity. To find I_r, I_L (Intensity through Resistance, Inductor, respectively) and U_C (Voltage for Capacitor), we use the Ohm's laws:

$$U_r = rI_r \quad , \quad U_L = L\frac{dI_L}{dt} \quad \text{and} \quad I_C = C\left(\frac{dU_C}{dt}\right).$$

We obtain $I_r + I_L = I^-$, $U^- = L\frac{dI_L}{dt}$ and $rI_r + U_C = U^-$ which are the Kirchoff's laws.

In vectorial form we obtain the implicit system :

$$Ax'(t) = Bx(t) + f(t) \quad ,$$

where

$$A = \begin{pmatrix} 000 \\ 0L0 \\ 000 \end{pmatrix}, \quad -B = \begin{pmatrix} 110 \\ 000 \\ r01 \end{pmatrix} , \quad f(t) = \begin{pmatrix} I^- \\ U^- \\ U^- \end{pmatrix} , \quad x(t) = \begin{pmatrix} I_r \\ I_L \\ U_C \end{pmatrix}.$$

2. The Generalized Liapounov Theorem

In this section we consider the Hilbert space H with the scalar product $< x, y >$, $(x, y) \in H^2$, and the linear bounded operator $V \in [H]$. The adjoint of V is denoted by V^*. Then, the spectra $\sigma(V)$ and $\sigma(V^*)$ are distributed symmetrically with respect to the real axis. So, if V is hermitian $(V = V^*)$, then the hermitian form $< Vx, x >$, $x \in H$ takes only real values. The spectrum $\sigma(V)$ of a hermitian operator V is a bounded closed set on the real axis. The least segment that contains $\sigma(V)$ will be denoted by $[\lambda_m(V), \lambda_M(V)]$.

As it is well known

$$\lambda_m(V) = \inf\left\{< Vx, x >: \| x \| = 1\right\} \quad ; \quad \lambda_M(V) = \sup\left\{< Vx, x >: \| x \| = 1\right\} \quad ; \quad \| V \| = \max\left\{\lambda_M(V), -\lambda_m(V)\right\} .$$

An operator $V \in [H]$ is said to be positive (non-negative) if its form $< Vx, x >$ is positive (non-negative) for any $x \neq 0$. Whenever V is non-negative has the norm $\| V \| = \lambda_M(V)$.

An operator V is said to be uniformly positive, and one writes $V \gg 0$, if its form $< Vx, x >$ is positive on the unit sphere $\left\{x \in H : \| x \| = 1\right\}$ in H. It means $\lambda_m(V) > 0$. Negative and uniformly negative operators $V \ll 0$ are defined analogously.

It is significant that every uniformly positive operator V permits to introduce in H a new scalar product $< x, y >_V = < Vx, y >$ with respect to which H remains a complete Hilbert space.

This occurs because the new norm $\| x \|_V = \sqrt{< Vx, x >}$ is topologically equivalent to the original in virtue of the estimates

$$\lambda_m(V) \| x \|^2 \leq \| x \|_V^2 \leq \| V \| . \| x \|^2 \quad ; \quad \lambda_m(V) > 0 \quad ; \quad \forall x \in H .$$

We will say that two different scalar products defined on the same set are topologically equivalent if the norms defined by them are topologically equivalent.

It can be asserted that any scalar product $< x, y >_1$ that is topologically equivalent to the original one can be obtained by means of a formula of the form

$$< x, y >_1 = < x, y >_V = < Vx, y > ,$$

where V is a uniformly positive operator.

We recall that the real part of a bounded operator $T \in [H]$ is the hermitian operator

$$\operatorname{Re} T \equiv \frac{1}{2}(T + T^*) \ ,$$

while its imaginary part is the hermitian operator

$$\operatorname{Im} T \equiv \frac{1}{2i}(T - T^*) .$$

So, that $T = \operatorname{Re} T + i \operatorname{Im} T$.

The Generalized Liapounov Theorem [2]. The spectrum of an operator $T \in [H]$ is in the interior of the left half-plane if and only, if there exists an uniformly positive operator V such that :

$$\mathrm{Re}(VT) \ll 0. \tag{5}$$

Moreover, if $\sigma(T)$ lies in the interior of the left half-plane, then for any $S \gg 0$ there exists an operator $V \gg 0$ such that the following Liapounov Equation holds

$$\mathrm{Re}(VT) = -S \Leftrightarrow VT + T^{*}V = -2S. \tag{6}$$

Thus, if $\mathrm{Re}\,T \ll 0$, the spectrum $\sigma(T)$ lies in the interior of the left half-plane [1].

3. The General Liapounov Theorem

In this article, we extend the Generalized Liapounov Theorem 2 [2] to the spectrum of the pencil $\lambda A - B$ of the bounded operators A and B in Hilbert space H corresponding to the degenerate differential system :

$$Ax'(t) = Bx(t) \ , \ t \ge 0 \quad . \tag{7}$$

Definition 3.1 [1]

The complex $\lambda \in \mathbb{C}$ is said to be a regular value of the pencil $\lambda A - B$, if the resolvent $(\lambda A - B)^{-1}$ exists and it is bounded. The set of all regular values is denoted by $\rho(A,B)$ and its complement $\sigma(A,B) = \mathbb{C} \setminus \rho(A,B)$ is called the spectrum of the pencil $\lambda A - B$. The set of all eigenvalues of the pencil $\lambda A - B$ is denoted by

$$\sigma_p(A,B) = \left\{ \lambda \in \mathbb{C} / \exists v \ne 0 : (\lambda A - B)v = 0 \right\}.$$

Theorem 3.2: Suppose that the spectrum $\sigma(A,B)$ of the pencil $\lambda A - B$ of bounded operators A and B is in the left half-plane. Then for any positive uniform operator $S \gg 0$ [2], there exists an operator $W \gg 0$ such that

$$A^{*}WB + B^{*}WA = -2S. \tag{8}$$

Proof. Suppose that $\sigma(A,B) \subset \left\{ \lambda : \mathrm{Re}\,\lambda < 0 \right\}$. Then $\lambda = 1$ is a regular value and the transformation of Cayley type : $T = -(A+B)(A-B)^{-1}$ is well defined and bounded. Now, using the conformal mapping $z = \varphi(\lambda) = \dfrac{\lambda+1}{\lambda-1}$, which transforms the imaginary axis ($\mathrm{Re}\,\lambda = 0$) into the unit circle ($|z| = 1$), we obtain:

$$T - zI = -(A+B)(A-B)^{-1} - \frac{\lambda+1}{\lambda-1}(A-B)(A-B)^{-1}$$

$$= \frac{-1}{\lambda-1}\left[(\lambda-1)(A+B) + (\lambda+1)(A-B) \right](A-B)^{-1}$$

$$= \frac{-2}{\lambda-1}(\lambda A - B)(A-B)^{-1}.$$

So, the operator $T - zI$ is not invertible if and only if the pencil $\lambda A - B$ is too. We conclude, that the spectrum of T equals $\sigma(T) = \sigma(-I,-T) = \varphi(\sigma(A,B))$. Therefore, $\sigma(T)$ is inside the unit disk. In virtue of the Theorem 2 [1] we have: For any operator $G \gg 0$ there exists an operator $W \gg 0$ such that :

$$T^{*}WT - W = -G. \tag{9}$$

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[1] In this case, $V = I \gg 0$, I is the identity operator.

[2] It means, that $S^{*} = S$ and there exist a constant $c > 0$ such that $\forall x \in H : \langle Sx, x \rangle \ge c\|x\|^2$.

It is equivalent to $\left[-(A+B)(A-B)^{-1}\right]^{*} W\left[-(A+B)(A-B)^{-1}\right]-W=-G$

$$\Leftrightarrow (A^{*}-B^{*})^{-1}(-A^{*}-B^{*})W(-A-B)(A-B)^{-1}-W=-G$$

$$\Leftrightarrow (-A^{*}-B^{*})W(-A-B)-(A^{*}-B^{*})W(A-B)=-(A^{*}-B^{*})G(A-B)$$

$$\Leftrightarrow 2(A^{*}WB+B^{*}WA)=-(A^{*}-B^{*})G(A-B)$$

$$\Leftrightarrow A^{*}WB+B^{*}WA=-2S\,;$$

where

$$-2S=-\frac{1}{2}(A^{*}-B^{*})G(A-B) \quad \text{or} \quad S=\frac{1}{4}(A^{*}-B^{*})G(A-B)\gg 0\cdot$$

In fact,

$$S^{*}=\frac{1}{4}(A^{*}-B^{*})G(A-B)=S\cdot$$

And , $\forall x\in H$ we have :

$$<Sx,x>=\frac{1}{4}<(A^{*}-B^{*})G(A-B)x,x> = \frac{1}{4}<Gy,y>\geq\frac{k}{4}\|y\|^{2}\;;\; y=(A-B)x\;;\;k>0\,.$$

But, $\|x\|^{2}=\|(A-B)^{-1}y\|^{2}\leq\|(A-B)^{-1}\|^{2}\|y\|^{2}\cdot$

Therefore $\|y\|^{2}\geq\dfrac{\|x\|^{2}}{\|(A-B)^{-1}\|^{2}}\cdot$

So ,

$$<Sx,x>\geq\frac{1}{4}\frac{k}{\|(A-B)^{-1}\|^{2}}\|x\|^{2}>0 \quad\cdot$$

Conclusion, $S\gg 0$. So, the relation (8) is proved.

Theorem 3.3: Suppose that $\lambda=1$ is a regular value for the pencil $\lambda A-B$ of bounded operators and there exists an operator $W\gg 0$ such that

$$A^{*}WB+B^{*}WA\ll 0\,. \qquad (10)$$

Then, the spectrum $\sigma(A,B)$ of the pencil $\lambda A-B$ is in the left half-plane.

Proof. If $\lambda=1$ is a regular value for the pencil $\lambda A-B$ then the operator $T=-(A+B)(A-B)^{-1}$ is bounded and the relation (10) becomes

$$A^{*}WB+B^{*}WA=-\frac{1}{2}(A^{*}-B^{*})G(A-B)\ll 0\cdot$$

Therefore, $G=W-T^{*}WT\gg 0$ (see the relation (9)). Using again Theorem 2 [1] the spectrum $\sigma(T)$ will be inside the unit disk. We conclude that

$$\sigma(A,B)=\varphi^{-1}(\sigma(T))\subset\left\{\lambda:\operatorname{Re}\lambda<0\right\}\ ,$$

where $\lambda=\varphi^{-1}(z)=\dfrac{z+1}{z-1}$ a conformal mapping and the Theorem 3.3 is proved.

Theorem 3.4: If the relation (8) is satisfying for the pair (W,S) of positive uniform operators, then , $\lambda=1$ is not an eigenvalue for the pencil $\lambda A-B$.

Proof. Suppose by absurd, that $\lambda=1$ is an eigenvalue. We design by $v\neq 0$ the corresponding eigenvector. Then , $(A-B)v=0$ or $Av=Bv$, and in this case the scalar product becomes :

$$\begin{aligned}<Sv,v>&=-\frac{1}{2}<A^{*}WBv,v>-\frac{1}{2}<B^{*}WAv,v>\\&=-\frac{1}{2}<WBv,Av>-\frac{1}{2}<WAv,Bv>\\&=-\frac{1}{2}<WBv,Bv>-\frac{1}{2}<WBv,Bv>\\&=-<WBv,Bv><0 \quad\cdot\end{aligned}$$

We obtain a contradiction, with the hypothesis $S\gg 0$, since $W\gg 0$ .Consequently the Theorem 3.4 is proved .

Remark 3.1: In the finite dimensional spaces, the hypothesis ($\lambda = 1$ is a regular value) in the Theorem 3.3 is evident in virtue of the Theorem 3.4.

4. Conclusion

In this paper we have presented the General Liapounov Theorem for some degenerate differential systems $Ax'(t) = Bx(t)$, in Hilbert spaces using the General Liapounov Equation and the Spectral Theory of the corresponding pencil of operators $\lambda A - B$.

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