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To study the different classes and inequalities involving multivalent functions

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Abstract
In this article we study multivalent functions in detail. We introduce and study some new subclasses of multivalent functions which are defined by differential subordination. The subclasses of these functions like starlike, convex, close to convex, spiral like, typical real functions. We obtain several properties like coefficient estimates, distortion bounds, radius of starlikeness, convexity and close to convex, extreme points, region of univalency, convex linear combination etc. We discuss different subclasses of multivalent that are holomorphic in nature.

Keywords: Multivalent functions, analytic functions, univalent functions

1. Introduction and Definitions

Let $p \in N$ {1, 2, 3,..} and $\mathcal{T}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

being analytic and p-valent in the open unit disk

$$\mathcal{U} = \{z: z \in C \text{ and } |z| < 1\}.$$

A function $f(z) \in \mathcal{T}(p)$ is said to be p-valently starlike in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \; (z \in \mathcal{U}). \tag{1.2}$$

A function $f(z) \in \mathcal{T}(p)$ is said to be p-valently convex in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \; (z \in \mathcal{U}). \tag{1.3}$$

Further, a function $f(z) \in \mathcal{T}(p)$ is said to p-valently close-to-convex in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0 \; (z \in \mathcal{U}). \tag{1.4}$$

(See, for details, [3], [5], and [13] for the above definitions.)
The following definitions of fractional calculus will be required in our present investigation:

Definition 1: (cf., [10] and [12]; see also [2]) Let a function $f(z)$ be analytic in a simply-connected region of the z-plane containing the origin. The fractional integral of order $\mu (\mu > 0)$ is defined by

$$D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_0^z f(\xi)(z-\xi)^{\mu-1} d\xi, \tag{1.5}$$

and the fractional derivative of order $\mu(0 \leq \mu < 1)$is defined by

$$D_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z f(\xi)(z-\xi)^{-\mu}d\xi, \quad (1.6)$$

where the multiplicity of $(z-\xi)^{\mu-1}$ involved in (1.5) and that of $(z-\xi)^{-\mu}$ in (1.6) are removed by requiring log $(z-\xi)$ to be real when $z-\xi > 0$.

Definition 2: (cf., [10] and [12]; see also [2]) Under the hypotheses of Definition 1, the fractional derivative of order $m+\mu$ $(m \in N_0 = N \cup \{0\}; 0 \leq \mu < 1)$ is defined by

$$D_z^{m+\mu}\{f(z)\} = \frac{d^m}{dz^m}D_z^{\mu}\{f(z)\}. \quad (1.7)$$

Now, by making use of the fractional derivative operator $D_z^{m+\mu}$, we define two important families $\mathcal{V}_\delta(p;\mu)$ and $\mathcal{W}_\delta(p;\mu)$ in $\mathcal{T}(p)$, where $\delta \in \mathcal{R}\{0\}, p \in \mathcal{N}$ and $0 \leq \mu < 1$.

Definition 3: Let $\delta \in \mathcal{R}\{0\}, p \in \mathcal{N}$and $0 \leq \mu 1$. Then a function $f(z) \in \mathcal{T}(p)$ is said to belong to $\mathcal{V}_\delta(p;\mu)$if it satisfies the inequality:

$$\left|\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right) - (p-\mu)^{\delta}\right| < (p-\mu)^{\delta} \ (z \in \mathcal{U}), \quad (1.8)$$

where the value of $\left(zD_z^{1+\mu}f(z)/D_z^{\mu}f(z)\right)^{\delta}$is taken its principal value.

Definition 4: Let $\delta \in \mathcal{R}\{0\}, p \in \mathcal{N}$ and $0 \leq \mu < 1$. Then a function $f(z) \in \mathcal{T}(p)$ is said to belong to $\mathcal{W}_\delta(p;\mu)$, if

$$\left|\left(z^{\mu-p}D_z^{\mu}f(z)\right)^{\delta} - \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^{\delta}\right| < \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^{\delta} \ (z \in \mathcal{U}), \quad (1.9)$$

by taking the principal value for $\left(z^{\mu-p}D_z^{\mu}f(z)\right)^{\delta}$.

Note that functions in $\mathcal{V}_1(p;0)$ are *p*-valently starlike in $\mathcal{U}$ (e.g. [9]). See, for examples, the papers involving the fractional calculus and/or certain inequalities, [1], [4], [6], [7], and [11]. In [6, 7] J, Irmak and cetin studied starlikeness and convexity for multivalent functions involving inequalities. In this paper we investigate various interesting properties for $\mathcal{V}_\delta(p;\mu)$ and $\mathcal{W}_\delta(p;\mu)$ associated with fractional calculus and also extend the results of Irmak and cetin ([6, 7]).

2. Main Results

Now, we mention the following result which is used in the sequal.

Lemma: Let *w*(*z*) be an analytic function in the unit disk $\mathcal{U}$ with *w*(0) = 0 and let 0 <*r* < 1. If $|w(z)|$ attains at z_0 its maximum value on the circle $|z| = r$, then

$$z_0 w'(z_0) = cw(z_0) \ (c \geq 1). \quad (2.1)$$

Making use of this lemma, we first give the following theorem:

Theorem 1: Let $\delta \in \mathcal{R}\{0\}, p \in \mathcal{N}$and $0 \leq \mu < 1$. If a function $f(z) \in \mathcal{T}(p)$ satisfies the inequality:

$$\Re e\left\{1 + z\left(\frac{D_z^{2+\mu}f(z)}{D_z^{1+\mu}f(z)} - \frac{D_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)\right\}\begin{Bmatrix} < 1/(2\delta) when\ \delta > 0 \\ > 1/(2\delta)\ when\ \delta < 0\end{Bmatrix} (z \in \mathcal{U}), \quad (2.2)$$

Then $f(z) \in \mathcal{V}_\delta(p;\mu)$.

Proof: First of all, Definition 1 readily provides us the following fractional derivative formula for a power function:

$$D_z^{\mu}\{z^k\} = \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)}z^{k-\mu} \ (k > -1; 0 \leq \mu < 1). \quad (2.3)$$

Define the function *w*(*z*) by

$$\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)^{\delta} = (p-\mu)^{\delta}[1 + w(z)] \ (z \in \mathcal{U}). \quad (2.4)$$

Then it follows from (2.3) that $w(z)$ is an analytic function in $\mathcal{U}$ and $w(0) = 0$. The logarithmically differentiation of (2.4) implies that

$$\mathcal{G}(z) := \left\{1 + z\left(\frac{D_z^{2+\mu}f(z)}{D_z^{1+\mu}f(z)} - \frac{D_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)\right\} = \frac{1}{\delta}\cdot\frac{zw'(z)}{1+w(z)}. \tag{2.5}$$

Now, suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1 \ (w(z_0) \ne -1).$$

Then, applying Jack's Lemma, we can write

$$z_0 w'(z_0) = cw(z_0)\ (c \ge 1)$$

and $w(z_0) = e^{i\theta}(\theta \ne \pi)$. Thus, from (2.5) we obtain

$$\begin{aligned}\Re e\{\mathcal{G}(z_0)\} &= \frac{1}{\delta}\Re e\left(\frac{z_0 w'(z_0)}{1+w(z_0)}\right)\\ &= \frac{c}{\delta}\Re e\left(\frac{e^{i\theta}}{1+e^{i\theta}}\right)\\ &= \frac{c}{2\delta}\begin{Bmatrix}\ge \frac{1}{2\delta}\ when\ \delta > 0\\ \le \frac{1}{2\delta} when\ \delta > 0\end{Bmatrix},\end{aligned} \tag{2.6}$$

where $\theta \ne \pi$ and $c \ge 1$. Therefore, (2.6) contradict our condition (2.2), and we conclude from the definition (2.4) that

$$\left|\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)^{\delta} - (p-\mu)^{\ \delta}\right| = (p-\mu)^{\ \delta}|w(z)| < (p-\mu)^{\ \delta},$$

which completes the proof of Theorem 1.

Theorem 2:2 Let $\delta \in \mathcal{R}\{0\}, p \in \mathcal{N}$ and $0\ \le \mu < 1$. If a funtion $f(z) \in \mathcal{T}(p)$satisfies the inequality:

$$\Re e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)\begin{Bmatrix}< p-\mu+1/(2\delta)\ when\ \delta > 0\\ > p-\mu+1/(2\delta)\ when\ \delta < 0\end{Bmatrix}\ (z \in \mathcal{U}), \tag{2.7}$$

then $f(z) \in \mathcal{W}_{\delta}(p;\mu)$.

Proof. Put

$$\left(z^{\mu-p}D_z^{\mu}f(z)\right)^{\delta} = \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\right)^{\delta}[1+w(z)]\ (z \in \mathcal{U}), \tag{2.8}$$

then, using the same technique as in the proof of Theorem 1, we get the desired result.
Many interesting results involving analytic and multivalent functions can be obtained by the use of Theorem 1 and Theorem 2 together with definitions (1.8) and (1.9) (respectivelly) and by choosing suitable values of δ, μ and p. Now, we are giving some of the important results for the analytic and geometric function theory (cf., [13]):
Letting $\delta = 1$ in Theorem 1, we have

Corollary 1: Let $p\ \in \mathcal{N}$ and $0 \le \mu < 1$. If a funtion $f(z) \in \mathcal{T}(p)$satisfies the inequality:

$$\Re e\left\{1 + z\left(\frac{D_z^{2+\mu}f(z)}{D_z^{1+\mu}f(z)} - \frac{D_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right)\right\} < \frac{1}{2}\ (z \in \mathcal{U}), \tag{2.9}$$

then $f(z) \in \mathcal{V}_1(p;\mu)$.
Making use of Theorem 2 and [2, Corollary 1], we obtain

Corollary 2: Let $p\ \in \mathcal{N}$ and $0 \le \mu < 1$. If a funtion $f(z) \in \mathcal{T}(p)$ satisfies the inequality:

$$\Re e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right) < p-\mu+\frac{1}{2}\ (z \in \mathcal{U}), \tag{2.10}$$

then $f(z) \in \mathcal{W}_1(p;\mu)$ and

$$\Re e \left\{\frac{D^{\mu-1} f(z)}{z^{p-\mu+1}}\right\} > \frac{\Gamma(p+1)}{\Gamma(p-\mu+2)} \left(1 + \sum_{k=1}^{\infty} \frac{2(p-\mu+1)(-1)^k}{p-\mu+k+1}\right) \ (z \in \mathcal{U}). \tag{2.11}$$

The estimate (2.11) is sharp in general.

Proof: If we take δ = 1 in Theorem 2, then the condition (2.10) implies $f(z) \in \mathcal{W}_1(p;\mu)$Further, from (1.9) it is easily shown that

$$\Re e \left\{\frac{D_z^{\mu} f(z)}{z^{p-\mu}}\right\} > 0.$$

Therefore, by virtue of [2, Corollary 1], we obtain the result.
Letting $\mu = 0$= 0 in Corollaries 1 and 2 (or, $\delta - 1 = \mu = 0$ in Theorems 1 and 2), we get already known results as indicated.

Corollary 3: (cf., [6, p. 457, Corollary 2]; see also [7, p. 74, Eq. (2.15), 2.2. Corollary]) Let $p \in N.$ If a funtion $f(z) \in \mathcal{T}(p)$ satisfies the inequality:

$$\Re e \left\{1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} < \frac{1}{2} \ (z \in \mathcal{U}), \tag{2.12}$$

then *f*(*z*) is p-valently starlike in $\mathcal{U}$.

Corollary 4: (cf., [6, p. 457, Corollary 1]) Let $p \in N$. If a funtion $f(z) \in \mathcal{T}(p)$ satisfies the inequality:

$$\Re e \left\{\frac{zf'(z)}{f(z)}\right\} < p + \frac{1}{2} \ (z \in \mathcal{U}), \tag{2.13}$$

Then $\Re e \left\{\frac{f(z)}{z^p}\right\} > 0 \ (z \in \mathcal{U}).$ (2.14)

Letting $\mu \to 1 -$in Corollaries 1 and 2 (or, $\mu \to 1 -$and $\delta = 1$ in Theorems 1 and 2), we have

Corollary 5: (cf., [6, p. 458, Corollary 4]; see also [7, p. 75, Eq. (2.17), 2.3. Corollary]) If a funtion $f(z) \in \mathcal{T}(p)$ satisfies the inequality:

$$\Re e \left\{1 + z\left(\frac{f'''(z)}{f''(z)} - \frac{f''(z)}{f'(z)}\right)\right\} < \frac{1}{2} \ (z \in \mathcal{U}; p \in \mathcal{N}\{1\}), \tag{2.15}$$

then *f*(*z*) is p-valently convex in *u*.

Corollary 6: w (cf., [2, Corollary 1) and [6, p. 458, Corollary 3]) Let $p \in \mathcal{N}$. If a funtion $f(z) \in \mathcal{T}(p)$satisfies the inequality:

$$\Re e \left\{\frac{zf''(z)}{f'(z)}\right\} < p - \frac{1}{2} \ (z \in \mathcal{U}), \tag{2.16}$$

then *f*(*z*) is p-valently close-to-convex in $\mathcal{U}$ and

$$\Re e \left\{\frac{f(z)}{z^p}\right\} > 1 + 2p \sum_{k=1}^{\infty} \frac{(-1)^k}{p+k} (z \in \mathcal{U}). \tag{2.17}$$

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