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The extension of measures

Baljodh Singh

Abstract
In this paper we have to show that if A is any algebra of subsets of a set X and if μ is a measure defined on A then there exist a σ- algebra A^* containing A and a measure μ^* defined on A^* such that $\mu^*(\mathrm{E}) = \mu(E)$ in A. In other words the measure μ can be extended to a measure on a σ-algebra A^* of subsets of X, which contains A.

Keywords: algebra, σ- algebra, measure, measurable set, finite and infinite measure

1. Introduction
1.1 Definition
Let X be any set and $\mathcal{A}$ be any non-empty subset of $P(X)$ the power set of X, $\mathcal{A}$ is said to be an algebra if

1. ϕ and $X \in \mathcal{A}$
2. $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$
3. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

Here it is easy to prove that if $\mathcal{A}$ is an algebra and A *and* B are in $\mathcal{A}$ then $A \cap B$ and A/B are also in $\mathcal{A}$ and also $A \Delta B = (A - B) \cup (B - A)$ belongs to $\mathcal{A}$.

1.2 Remark: $\mathcal{A}$ is closed under finite unions and intersections that if $A_1, A_2, \dots .. A_{\mathrm{n}} \in \mathcal{A}$ then

$$\begin{cases} \bigcup_{i=1}^{n} A_i \in \mathcal{A} \\ \bigcap_{i=1}^{n} A_i \in \mathcal{A} \end{cases}$$

Example: 1. If $X = [0, 1)$, then the set $\mathcal{A}$ consisting of ϕ and all unions $A = \bigcup_{i=1}^{n} [a_i, b_i)$ where $0 \le a_i < b_i \le a_{i+1} \le 1$ is an algebra.
2. If X be a finite set then consider the set
E = $\{A \in \mathrm{P(X)}/$ A is finite or A^c is finite$\}$ then E is an algebra. It can be easily verified that E is closed under finite unions and finite intersections.
3. Obviously for any set X the power set P(X) and E= $\{ \phi, X \}$ are $\sigma - al$gebras in X and P(X) is called the largest and E = $\{ \phi, X \}$ is the smallest $\sigma - al$gebra.

1.3 Definition: If B be an arbitrary subset of X, define $\mu^*(\mathrm{B})$ = Inf $\sum_{i=1}^{\infty} \mu (E_i)$ where the infimum is extended over all sequences $\{E_i\}$ of sets in A such that $B \subseteq \bigcup_{i=1}^{\infty} E_i$. Generally the function μ^* defined above is called outer measure generated by μ.

1.4 Lemma: The function μ^* defined above satisfy the following
(1) $\mu^*(\phi) = 0$
(2) $\mu^*(A) \ge 0$ for any A ⊆ X.
(3) If A ⊆ B, then $\mu^*(A) \le \mu^*(B)$.
(4) If A ∈ $\mathcal{A}$ then $\mu^*(A) = \mu (A)$.
(5) If (E_i) be a sequence of subsets of X, then $\mu^*(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} \mu^* (E_i)$.

Proof: Statements (1), (2) and (3) are immediate consequences of the definition 1.1.
(4) Since $\{A, \phi, \phi, \ldots\}$ is a countable collection of sets in $\mathcal{A}$ whose union contains A, then it follows that $\mu^*(A) \leq \mu(A) + 0 + 0 + \ldots\ldots\ldots = \mu(A)$. …………….. (1)
Conversely, If (E_i) be any sequence of subsets from $\mathcal{A}$ with $A \subseteq \cup E_i$, then $A = \cup (A \cap E_i)$. Since μ is a measure on $\mathcal{A}$, then

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(A \cap E_i) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Follows that $\mu(A) \leq \mu^*(A)$ …………………. (2)

Hence from (1) and (2) we have $\mu^*(A) = \mu(A)$.
(5) Let $\varepsilon > 0$ be arbitrary and for each n chose a sequence (E_{nk}) of the sets in $\mathcal{A}$ such that $B_n \subseteq \bigcup_{k=1}^{\infty} E_{nk}$ and $\sum_{k=1}^{\infty} \mu(E_{nk}) \leq \mu^*(B_n) + \frac{\varepsilon}{2^n}$. Since $\{E_{nk} : n, k \in \mathrm{N}\}$ is a countable collection from $\mathcal{A}$ whose union contains $\cup B_n$ it follows from the definition of μ^* that $\mu^*(\bigcup_{i=1}^{n} B_n) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(E_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(B_n) + \varepsilon$. Since ε is arbitrary, hence proved.

1.5 Remark: The property (5) is referred by saying that μ^* in countably sub-additive. Although μ^* has the advantage that it is defined for arbitrary subsets of X, it has the defect that it is not necessarily countably additive. Hence we are willing to restrict μ^* to a smaller $\sigma-$algebra provided we can find one containing $\mathcal{A}$ and over which μ^* has the property of countable additivity. There is a remarkable condition due to Caratheodory which provides the desired restriction of the domain of μ^*.

1.6 Definition: A subset E of X is said to be to be μ^*-measurable if

$$\mu^*(\mathrm{A}) = \mu^*(\mathrm{A} \cap \mathrm{E}) + \mu^*(\mathrm{A} \backslash \mathrm{E}) \quad \text{………….. (A)}$$

for all sub sets A of X. The collection of all μ^*-measurable sets is denoted by $\mathcal{A}^*$. Condition (A) indicated an addtivity property on μ^*

1.7 Caratheodory Extension theorem: The collection $\mathcal{A}^*$ of all μ^*-measurable sets is a $\sigma-$algebra containing $\mathcal{A}$. Moreover if (E_n) is a disjoint sequence in $\mathcal{A}^*$, then

$$\mu^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n).$$

Proof: It is clear that ϕ and X are μ^*-measurable sets and if $\mathrm{E} \in \mathcal{A}^*$ then its complement X \ E belongs to $\mathcal{A}^*$. Next we shall show that $\mathcal{A}^*$ is closed under intersections. Indeed suppose that E and F are μ^*-measurable. Then for any $\mathrm{A} \subseteq \mathrm{X}$ and $\mathrm{E} \in \mathcal{A}^*$, we have $\mu^*(\mathrm{A} \cap \mathrm{F}) = \mu^*(\mathrm{A} \cap \mathrm{F} \cap \mathrm{E}) + \mu^*((\mathrm{A} \cap \mathrm{F})\backslash \mathrm{E})$
Since $\mathrm{F} \in \mathcal{A}^*$ then then $\mu^*(\mathrm{A}) = \mu^*(\mathrm{A} \cap \mathrm{F}) + \mu^*(\mathrm{A}\backslash \mathrm{F})$.
Let B = A\ (E∩ F), then it is readily seen that B∩ F = A∩ F\E and B\F = A\F; since $\mathrm{F} \in \mathcal{A}^*$ it follows that $\mu^*((\mathrm{A}(\mathrm{E} \cap \mathrm{F})) = \mu^*((\mathrm{A} \cap \mathrm{F})\backslash \mathrm{E}) + \mu^*(\mathrm{A}\backslash \mathrm{F})$
Combining these three relations we get $\mu^*(\mathrm{A}) = \mu^*(\mathrm{A} \cap \mathrm{E} \cap \mathrm{F}) + \mu^*(\mathrm{A}\backslash(\mathrm{E} \cap \mathrm{F}))$.
Which shows that E ∩ F belongs to $\mathcal{A}^*$. Since $\mathcal{A}^*$ is closed under intersection and complementation, it follows that $\mathcal{A}^*$ is an algebra.
Suppose that E, $\mathrm{F} \in \mathcal{A}^*$ and $\mathrm{E} \cap \mathrm{F} = \emptyset$. If we take A to be A ∩ (E ∪ F) in (A) we get

$$\mu^*(\mathrm{A} \cap (\mathrm{E} \cup \mathrm{F})) = \mu^*(\mathrm{A} \cap \mathrm{E}) + \mu^*(\mathrm{A} \cap \mathrm{F})$$

For A= X, this relation implies that $\mu^*$is additive on $\mathcal{A}^*$.
We shall now show that $\mathcal{A}^*$ is a $\sigma-$algebra and that μ^* is countably additive on $\mathcal{A}^*$. Let (E_i) be a disjoint sequence in $\mathcal{A}^*$ and let E = $\cup E_k$ if A be any subset of X then
$\mu^*(\mathrm{A}) = \mu^*(\mathrm{A} \cap F_n) + \mu^*(\mathrm{A}\backslash F_n) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k)$ - $\mu^*(\mathrm{A}\backslash F_n)$.
Since $F_n \subseteq \mathrm{E}$, then A\E, then A\E ⊆A\F_n and letting n → ∞ the above relation yields
$\sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(\mathrm{A}\backslash E) \leq \mu^*(\mathrm{A})$ On the other hand it follows from Lemma 1.5 (5) that $\mu^*(\mathrm{A} \cap E) \leq \sum_{k=1}^{\infty} \mu^*(A \cap E_k)$, $\mu^*(\mathrm{A}) \leq \mu^*(\mathrm{A} \cap E) + \mu^*(\mathrm{A}\backslash E)$ on combining these inequalities we get that
$\mu^*(\mathrm{A}) = \mu^*(\mathrm{A} \cap E) + \mu^*(\mathrm{A}\backslash E) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(\mathrm{A}\backslash E)$.
In particular this shows that E = $\bigcup_{k=1}^{\infty} E_k$ is $\mu^* -$ measurable. On taking A= E, we obtain $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$.
Now it remains to prove that A ⊆ A^*. It was proved in lemma 1.5(4) that if $\mathrm{E} \in \mathcal{A}$, then $\mu^*(E) = \mu(E)$, but we need to show that E is $\mu^* -$ measurable. Let A be an arbitrary subset of X; it follows from 1.5 (5) that $\mu^*(\mathrm{A}) \leq \mu^*(A \cap E) + \mu^*(A\backslash E)$. To establish the opposite inequality, let $\varepsilon > 0$ be arbitrary and let (F_n) be a sequence in $\mathcal{A}$ such that
$\mathrm{A} \subseteq \cup F_n$ and $\sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(\mathrm{A}) + \varepsilon$.
Since A ∩ E ⊆∪ (F_n ∩ E) and A\E ⊆ ∪ (F_n\E), it follows that

$\mu^*(A \cap E) \leq \sum_{n=1}^{\infty} \mu(F_n \cap \mathrm{E})$, $\mu^*(A\backslash E) \leq \sum_{n=1}^{\infty} \mu(F_n\backslash \mathrm{E})$. Hence we have

$\mu^*(A \cap E) + \mu^*(A \setminus E) \leq \sum_{n=1}^{\infty}\{\mu(F_n \cap \text{E}) + \mu(F_n \setminus \text{E})\} = \sum_{n=1}^{\infty} \mu(F_n) \leq \mu^*(\text{A}) + \varepsilon.$

Since ε is arbitrary, the desired inequality is established and the set E belongs to $\mathcal{A}^*$.
The Caratheodory Extension Theorem shows that a measure μ on an algebra $\mathcal{A}$ can always be extended to a measure μ^* on a σ −algebra $\mathcal{A}^*$ containing $\mathcal{A}$. The σ-algebra $\mathcal{A}^*$obtained in this way is automatically complete in the sense that if E ∈ $\mathcal{A}^*$ with μ^*(E) = 0 and if B ⊆ E, then B ∈ $\mathcal{A}^*$and μ^*(B) = 0. To prove this let A be an arbitrary subset of X and employ Lemma 1.5(3) to observe that μ^*(A) = μ^*(E) + μ^*(A) ≥ μ^*(A ∩ B) +μ^*(A\B); and as before the inequality μ^*(A) ≤ μ^*(A ∩ B) + μ^*(A\B) follows from Lemma 1.5(5). Hence B is μ^*-measurable and 0 ≤ μ^* (B) ≤ μ^*(E) ≤ 0. We shall now show that in the case that μ is a σ-finite measure, it has a unique extension to a measure on $\mathcal{A}^*$.

1.8 Hahn Extension Theorem: Suppose that μ is a σ- finite measure on an algebra $\mathcal{A}$. Then there exist a unique extension of μ to a measure on $\mathcal{A}^*$.

Proof: The fact that μ^* gives a measure on $\mathcal{A}^*$ proved in theorem 1.7 even without the σ- finiteness assumption. To establish the uniqueness, let v be a measure on $\mathcal{A}^*$ which agrees with μ on $\mathcal{A}$.
First suppose that μ and therefore μ^* and v are finite measures. Let E be any set in $\mathcal{A}^*$ and let (E_n) be a sequence in $\mathcal{A}$ such that E ⊆ ∪ E_n. Since v be a measure and agrees with μ on $\mathcal{A}$ we have $v(E) \leq v\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} v(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$. Therefore $v(E) \leq \mu^*$(E) for any E ∈ $\mathcal{A}^*$. Since μ^* and v are additive. μ^*(E) + μ^*(X\E) + $v(E) + v$(X\E). Since the terms on the right hand side are finite and not greater than the corresponding terms on the left hand side, we infer that μ^*(E) + $v(E)$ for all E ∈ $\mathcal{A}^*$. This establishes the uniqueness when μ is a finite measure. Suppose that μ is σ-finite and let (F_n) be an increasing sequence of sets in $\mathcal{A}$ with $\mu(F_n) < \infty$ and X = ∪ F_n. Then from the proceeding $\mu^*(E \cap F_n) = v(E \cap F_n)$ for each E ∈ $\mathcal{A}^*$.
Therefore μ^*(E) = lim $\mu^*(E \cap F_n)$ = lim $v(E \cap F_n)$ = v(E), hence μ and μ^* agrees on $\mathcal{A}^*$.

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