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Integration and differentiation involving the laguerre polynomial of two variable $\mathcal{L}_n$ (x, y)

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Abstract
In this paper we obtain integration and partial differentiation involving the generalized associated Laguerre Polynomial of two variables $\mathcal{L}_n^{(\alpha)}$ (x, y) which are is closely related to generalized Lagguerre Polynomial of Dattoli *et al*. These results provide useful extensions of well-known results of Lagguere Polynomials $\mathcal{L}_n$(x)

Keywords: Laguerre Polynomials, Dattoli *et al.*, recurrence relation

1. Introduction

Two variable one index Laguerre polynomials have been given by Dattoli *et al.* [1, 3].
Two variable one index Laguerre polynomials defined as

$$\mathcal{L}_n(x, y) = n! \sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!\,(r!)^2} \qquad \ldots (1.1)$$

$\mathcal{L}_n$ (x, y) are linked to the ordinary Laguerre polynomials L_n (x) by

$$\mathcal{L}_n(x, 1) = L_n(x) \qquad \ldots (1.2)$$

$$\mathcal{L}_n(x, y) = y^n L_n\left(\frac{x}{y}\right). \qquad \ldots (1.3)$$

A generalization of (1.1) provided by the following definition of the generalized associated Laguerre polyniomials

$$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{r=0}^{n} \frac{(-1)^r (1+\alpha)_n\, y^{n-r} x^r}{(n-r)!\, r!\, (1+\alpha)_r} \qquad \ldots (1.4)$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (\alpha+n)!\, y^{n-r} x^r}{r!\,(n-r)!\,(\alpha+r)!},$$

and the generating function, we get

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n = (1-yt)^{-\alpha-1} \left(\frac{-xt}{1-yt}\right), \qquad \ldots (1.5)$$

Now using expansion on R.H.S. in (1.5) and after some calculation, we get

$$x^n = \sum_{r=0}^{n} \frac{(-1)^r \, n!\,(1+\alpha)_n \, y^{n-r}}{(n-r)!\,(1+\alpha)_r} \mathcal{L}_r^{(\alpha)}(\mathbf{x}, y) \qquad \ldots (1.6)$$

In this paper we shall give some basic relation and properties then obtain integral and differentiation involving the generalized associated Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x, y)$

Integral Involving Laguerre polynomials

I. To show $\int_0^\infty e^{-st} \mathcal{L}_n(xt, y)dt = \frac{y^n}{s}\left(1 - \frac{x}{sy}\right)^n$ … (2.1)

Proof: Replace x by xt in $\mathcal{L}_n$ (x, y) and multiply by e^{-st} then integrating with respect to t with in Limits 0 to ∞

$$\int_0^\infty e^{-st} \mathcal{L}_n(xt, y)dt = \int_0^\infty e^{-st} y^n$$

Put $\frac{xt}{y} = z$ in R.H.S. and using [5; P.216 (10)], we get required result (2.1)

II. To show

$$\int_0^\infty (x-t)^m \mathcal{L}_n(t, y)\,dt = \frac{m!\,n!}{(m+n+1)!} \mathcal{L}_n^{(m+1)}(x, y) \qquad \ldots (2.2)$$

Proof

$$\int_0^\infty (x-t)^m \mathcal{L}_n(t, y)\,dt = n!\sum_{r=0}^{n} \frac{(-1)^r y^{n-r}}{(n-r)!\,(r!)^2} \int_0^x (x-t)^m t^r\,dt$$

On putting t = xy, we get

$$= n!\sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^{m+r+1}}{(n-r)!\,(r!)^2} \int_0^1 (1-u)^m u^r\,du$$

$$= n!\sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^{m+r+1}\, m!}{(n-r)!\,(r!)\,(m+r\;\;1)!}$$

$$= \frac{n!\,m!\,x^{m+1}}{(m+n+1)!} \mathcal{L}_n^{(m+1)}(x, y)$$

which is a required result (2.2)

III If n is an odd in tiger then

$$\int_0^t \mathcal{L}_n[x(t-x), y]\,dx = \frac{(-1)^n H_{2n+1}\left(t/2, y\right)}{2^{2n}\left(3/2\right)_n}$$

Proof: Taking L.H.S. of (2.3)

$$\int_0^t \mathcal{L}_n(x(t-x), y\,dx = \sum_{r=0}^{n} \frac{(-1)^r n!\,x^r (t-x)^r y^{n-r}}{(n-r)!\,(r!)^2} dx$$

put x = tu and using same procedure as [6; P. 153], we get

$$= \sum \frac{(-1)^{n-r}\, n!\, y^r\, t^{2(n-r)+1}}{r!\,(2n-2r+1)!}$$

$$= \frac{(-1)^n H_{2n+1}\left(t/2, y\right)}{2^{2n}\left(3/2\right)_n}$$

which is a required result (2.3)

IV. If $n \geq 1$, then

$$\int_x^\infty e^{-y}\, [\mathcal{L}_n (y, z) - (z-1)\, \mathcal{L}_{n-1} (y, z)]\, dy$$

$$= e^{-x}\, [\mathcal{L}_n (x, z) - z\, \mathcal{L}_{n-1} (x, z)] \qquad \ldots (2.4)$$

Proof:

Taking L.H.S. $= \int_x^\infty e^{-y}\, [\mathcal{L}_n (y, z) - (z-1)\, \mathcal{L}_{n-1} (y, z)]\, dy$

$= I_1 + I_2$ … (2.5)

where $I_1 = \int_x^\infty e^{-y}\, \mathcal{L}_n (y, z)\, dy$

And $I_2 = -\int_x^\infty e^{-y}\, (z-1)\, \mathcal{L}_{n-1} (y, z)\, dy$

$$\text{since } I_1 = e^{-x} \mathcal{L}_n (x, z) + \int_x^\infty e^{-y} \frac{\partial}{\partial y} \mathcal{L}_n (y, z)\, dy \qquad \ldots (2.6)$$

Now using differential recurrence relation for $\mathcal{L}_n (x, y)$

$$\frac{\partial}{\partial x} \mathcal{L}_n (x, y) = y \frac{\partial}{\partial x} \mathcal{L}_{n-1} (x, y) - \mathcal{L}_{n-1} (x, y);\ n \geq 1 \qquad \ldots (2.7)$$

using (2.6) and (2.7) and after integrating we get

$I_1 = e^{-x}\, [\mathcal{L}_n (x, z) - z\, \mathcal{L}_{n-1} (x, z)] - I_2$

Now using (2.5), then we get required result (2.4)

Partial differentiation of $\mathcal{L}_n (x,y)$

Theorem – 3: If k be a positive integer then

$$\mathcal{L}_n^{(k)} (x, y) = (-1)^k \frac{\partial}{\partial x^k} \mathcal{L}_{n+k} (x, y)] \qquad \ldots (3.1)$$

Proof: By Definition of $\mathcal{L}_n (x, y)$

$$\mathcal{L}_{n+k} (x, y) = \sum_{r=0}^{n+k} \frac{(-1)^r\, (n+k)!\, x^r\, y^{n+k-r}}{(r!)^2\, (n+k-r)!}$$

So that

$$(-1)^k \frac{\partial^k}{\partial x^k} \mathcal{L}_{n+k} (x, y) = \sum_{r=0}^{n+k} \frac{(-1)^{r+k}\, (n+k)!\, x^{r-k}\, y^{n+k-r}}{(r!)\, (n+k-r)!\, (r-k)!} \qquad \ldots (3.2)$$

since $\frac{\partial^k}{\partial x^k}(x^r) = \frac{r!}{(r-k)!} x^{r-k}$, if the series starts from

r = 0; then $x^0 = 1$ and $\frac{\partial^k}{\partial x^k}$ can-not be operated.

Thus $\frac{\partial^k}{\partial x^k}$ can be operated when power of $x \geq k$

i.e. the series must start from r = k,
i.e. on changing the summation by substituting
s = r – k in (3.2), we get

$$(-1)^k \frac{\partial^k}{\partial x^k} \mathcal{L}_{n+k}(x, y) = \sum_{s=0}^{n} \frac{(-1)^{s+2k} (n+k)!\, x^s\, y^{n-s}}{(s+k)!\,(n-s)!\, s!} = \mathcal{L}_n^{(k)}(x, y)$$

which is a required result (3.1).

Special cases

I. For x = y = 1, then (2.1) reduces to known result [5; P.216 (10)]
For y = 1, then (2.2) reduces to a known result [6; P.160 (9)]
For y = 1, then (2.3) reduces to a known result [4; P. 149 (5)]
For z = 1, then (2.4) reduces to a known result [4; P. 149 (4)]

II. For y = 1, then (3.1) reduces to a known result [4; P. 151 (7.5)]

Special cases from I and II are known formulae for integral and partial differentiation for ordinary Laguerre Polynomial $\mathcal{L}_n(x)$.

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