International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2017; 2(6): 99-102

www.mathsjournal.com
Received: 09-09-2017
Accepted: 10-10-2017

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Characteristic equation of square matrix of order 4 and 5 in terms of determinants of its sub matrices

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Abstract
The method to find the characteristic equation of a square matrix of any order is to solve the equation det. $(A - \lambda I) = 0$ i.e. $|A - \lambda I| = 0$ which takes so much time when n = 4,5,…..& so on. This paper is an attempt to simplify the method to find the characteristic equation of a square matrix A of order 4 and 5 by taking the determinants of its all submatrices starting from 1x1 upto nxn.

Keywords: square matrix, sub matrices

Introduction
Definitions
Square matrix: A rectangular matrix $[a_{ij}]_{mxn}$ of order m x n is said to be square matrix if m = n
Determinant: Determinant of a square matrix is a number associated to it.
Characteristic equation: If $A=[a_{ij}]_{nxn}$ is square matrix of order n then equation $|A-\lambda I| = 0$ is its characteristic equation

Result
If $A= [a_{ij}]_{nxn}$ is square matrix of order n then its characteristic equation is given by $|A-\lambda I| = 0$

i.e. $(-1)^n\lambda^n + (-1)^{n-1}\sum_{i=1}^{n}|a_{ii}|\lambda^{n-1} + (-1)^{n-2}\sum_{i=1}^{n-1}\sum_{j=2}^{n}(i < j)\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}\lambda^{n-2}$

$+ (-1)^{n-3}\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n}(i < j < k)\begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}\lambda^{n-3}$

$+ (-1)^{n-4}\lambda^{n-4}\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n}(i < j < k < l)\begin{vmatrix} a_{ii} & a_{ij} & a_{ik} & a_{il} \\ a_{ji} & a_{jj} & a_{jk} & a_{jl} \\ a_{ki} & a_{kj} & a_{kk} & a_{kl} \\ a_{li} & a_{lj} & a_{lk} & a_{ll} \end{vmatrix} + \ldots + (-1)^0\lambda^0|A| = 0$

Corllary 1: For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$,

the characteristic equation is
$(-1)^4\lambda^4 + (-1)^{4-1}\sum_{i=1}^{4}|a_{ii}|\lambda^{4-1} + (-1)^{4-2}\sum_{i=1}^{4-1}\sum_{j=2}^{4}(i < j)\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}\lambda^{4-2} + (-1)^{4-3}\sum_{i=1}^{4-2}\sum_{j=2}^{4-1}\sum_{k=3}^{4}(i < j < k)\begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}\lambda^{4-3} + (-1)^0\lambda^0|A| = 0$

Or $(-1)^4\lambda^4 + (-1)^3(a_{11} + a_{22} + a_{33} + a_{44})\lambda^3$.

$+(-1)^2(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix})\lambda^2$

$+ (-1)^1 (\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix})\lambda^1$

$+ (-1)^0 \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \lambda^0 = 0$

Example: Take $A = \begin{bmatrix} 3 & -3 & 5 & 1 \\ 1 & -2 & 3 & 0 \\ 4 & 5 & 2 & 9 \\ -2 & -5 & -1 & 1 \end{bmatrix}$

Then characteristic equation of A is given by $|A - \lambda I| = 0$

or $\begin{vmatrix} 3-\lambda & -3 & 5 & 1 \\ 1 & -2-\lambda & 3 & 0 \\ 4 & 5 & 2-\lambda & 9 \\ -2 & -5 & -1 & 1-\lambda \end{vmatrix} = 0$

or $(3-\lambda) \begin{vmatrix} -2-\lambda & 3 & 0 \\ 5 & 2-\lambda & 9 \\ -5 & -1 & 1-\lambda \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 & 0 \\ 4 & 2-\lambda & 9 \\ -2 & -1 & 1-\lambda \end{vmatrix} + 5 \begin{vmatrix} 1 & -2-\lambda & 0 \\ 4 & 5 & 9 \\ -2 & -5 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -2-\lambda & 3 \\ 4 & 5 & 2-\lambda \\ -2 & -5 & -1 \end{vmatrix} = 0$

or $(3-\lambda)(-\lambda^3 + \lambda^2 + 10\lambda - 172) + 3(\lambda^2 + 9\lambda - 55) + 5(-4\lambda^2 + 9\lambda + 94) - 1(-2\lambda^2 - 9\lambda - 25) = 0$
or $\lambda^4 - 4\lambda^3 - 22\lambda^2 + 283\lambda - 186 = 0$
Using the above result

$(-1)^4 \lambda^4 + (-1)^3 (3+(-2)+2+1)\lambda^3$

$+(-1)^2(\begin{vmatrix} 3 & -3 \\ 1 & -2 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ -5 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 9 \\ -1 & 1 \end{vmatrix})\lambda^2$

$+ (-1)^1 (\begin{vmatrix} 3 & -3 & 5 \\ 1 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix} + \begin{vmatrix} 3 & -3 & 1 \\ 1 & -2 & 0 \\ -2 & -5 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 5 & 1 \\ 4 & 2 & 9 \\ -2 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 3 & 0 \\ 5 & 2 & 9 \\ -5 & -1 & 1 \end{vmatrix})\lambda^1$

$+ \begin{vmatrix} 3 & -3 & 5 & 1 \\ 1 & -2 & 3 & 0 \\ 4 & 5 & 2 & 9 \\ -2 & -5 & -1 & 1 \end{vmatrix} = 0$

We get the same result $\lambda^4 - 4\lambda^3 - 22\lambda^2 + 283\lambda - 186 = 0$

Corllary 2: For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$,

the characteristic equation is
$(-1)^5\lambda^5 + (-1)^{5-1}\sum_{i=1}^{5} |a_{ii}| \lambda^{5-1} + (-1)^{5-2}\sum_{i=1}^{5-1} \sum_{j=2}^{5}(i < j) \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \lambda^{5-2}$

$+ (-1)^{5-3}\sum_{i=1}^{5-2} \sum_{j=2}^{5-1} \sum_{k=3}^{5}(i < j < k) \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \lambda^{5-3}$

$+(-1)^{5-4}\sum_{i=1}^{5-3} \sum_{j=2}^{5-2} \sum_{k=3}^{5-1} \sum_{l=4}^{5}(i < j < k < l) \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} & a_{il} \\ a_{ji} & a_{jj} & a_{jk} & a_{jl} \\ a_{ki} & a_{kj} & a_{kk} & a_{kl} \\ a_{li} & a_{lj} & a_{lk} & a_{ll} \end{vmatrix} \lambda^{5-4} + (-1)^{5-5}\lambda^{5-5}|A| = 0$

Or $(-1)^5 \lambda^5 + (-1)^4 (a_{11} + a_{22} + a_{33} + a_{44} + a_{55})\lambda^4$

$$+(-1)^3\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{15} \\ a_{51} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix}+\right.$$

$$\left.\begin{vmatrix} a_{22} & a_{25} \\ a_{52} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{33} & a_{35} \\ a_{53} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix}\right)\lambda^3$$

$$+(-1)^2\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{51} & a_{52} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \\ a_{51} & a_{53} & a_{55} \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} a_{11} & a_{14} & a_{15} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} & a_{25} \\ a_{32} & a_{33} & a_{35} \\ a_{52} & a_{53} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{24} & a_{25} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{33} & a_{34} & a_{35} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}\right)\lambda^2$$

$$+(-1)^1\left(\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{51} & a_{52} & a_{53} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{24} & a_{25} \\ a_{41} & a_{42} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{54} & a_{55} \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} a_{11} & a_{13} & a_{14} & a_{15} \\ a_{31} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{53} & a_{54} & a_{55} \end{vmatrix}+\begin{vmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ a_{32} & a_{33} & a_{34} & a_{35} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}\right)\lambda^1+(-1)^0\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}\lambda^0=0$$

Example: Take $A = \begin{bmatrix} -2 & 3 & 5 & 6 & -1 \\ 9 & 7 & -1 & 0 & 3 \\ -4 & -2 & 1 & 3 & 2 \\ 5 & 9 & 3 & 2 & 3 \\ 4 & 5 & 6 & 1 & 2 \end{bmatrix}$

Then characteristic equation of A is given by $|A - \lambda I| = 0$

Or $\begin{vmatrix} -2-\lambda & 3 & 5 & 6 & -1 \\ 9 & 7-\lambda & -1 & 0 & 3 \\ -4 & -2 & 1-\lambda & 3 & 2 \\ 5 & 9 & 3 & 2-\lambda & 3 \\ 4 & 5 & 6 & 1 & 2-\lambda \end{vmatrix} = 0$

Or $(-2-\lambda)\begin{vmatrix} 7-\lambda & -1 & 0 & 3 \\ -2 & 1-\lambda & 3 & 2 \\ 9 & 3 & 2-\lambda & 3 \\ 5 & 6 & 1 & 2-\lambda \end{vmatrix} -3\begin{vmatrix} 9 & -1 & 0 & 3 \\ -4 & 1-\lambda & 3 & 2 \\ 5 & 3 & 2-\lambda & 3 \\ 4 & 6 & 1 & 2-\lambda \end{vmatrix} +5\begin{vmatrix} 9 & 7-\lambda & 0 & 3 \\ -4 & -2 & 3 & 2 \\ 5 & 9 & 2-\lambda & 3 \\ 4 & 5 & 1 & 2-\lambda \end{vmatrix} -6\begin{vmatrix} 9 & 7-\lambda & -1 & 3 \\ -4 & -2 & 1-\lambda & 2 \\ 5 & 9 & 3 & 3 \\ 4 & 5 & 6 & 2-\lambda \end{vmatrix}$

$+(-1)\begin{vmatrix} 9 & 7-\lambda & -1 & 0 \\ -4 & -2 & 1-\lambda & 3 \\ 5 & 9 & 3 & 2-\lambda \\ 4 & 5 & 6 & 1 \end{vmatrix} = 0$

Or $(-2-\lambda)(\lambda^4-12\lambda^3+2\lambda^2+192\lambda^1-288)-3(-9\lambda^3+29\lambda^2+276\lambda^1-112)+5(-4\lambda^3+49\lambda^2+164\lambda^1+16)-6(-5\lambda^3-31\lambda^2+84\lambda^1+720)-1(4\lambda^3-14\lambda^2+160\lambda^1+608) = 0$

Or $-\lambda^5 + 10\lambda^4+55\lambda^3+162\lambda^2-768\lambda^1-3936 = 0$

Use the above result

$(-1)^5\lambda^5 + (-1)^4(-2+7+1+2+2)\lambda^4$

$$+(-1)^3\left(\begin{vmatrix} -2 & 3 \\ 9 & 7 \end{vmatrix}+\begin{vmatrix} -2 & 5 \\ -4 & 1 \end{vmatrix}+\begin{vmatrix} -2 & 6 \\ 5 & 2 \end{vmatrix}+\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix}+\begin{vmatrix} 7 & -1 \\ -2 & 1 \end{vmatrix}+\begin{vmatrix} 7 & 0 \\ 9 & 2 \end{vmatrix}+\begin{vmatrix} 7 & 3 \\ 5 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix}+\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}\right)\lambda^3$$

$$+(-1)^2\left(\begin{vmatrix} -2 & 3 & 5 \\ 9 & 7 & -1 \\ -4 & -2 & 1 \end{vmatrix}+\begin{vmatrix} -2 & 3 & 6 \\ 9 & 7 & 0 \\ 5 & 9 & 2 \end{vmatrix}+\begin{vmatrix} -2 & 3 & -1 \\ 9 & 7 & 3 \\ 4 & 5 & 2 \end{vmatrix}+\begin{vmatrix} -2 & 5 & 6 \\ -4 & 1 & 3 \\ 5 & 3 & 2 \end{vmatrix}+\begin{vmatrix} -2 & 5 & -1 \\ -4 & 1 & 2 \\ 4 & 6 & 2 \end{vmatrix}\right.$$

$$\left.+\begin{vmatrix} -2 & 6 & -1 \\ 5 & 2 & 3 \\ 4 & 1 & 2 \end{vmatrix}+\begin{vmatrix} 7 & -1 & 0 \\ -2 & 1 & 3 \\ 9 & 3 & 2 \end{vmatrix}+\begin{vmatrix} 7 & -1 & 3 \\ -2 & 1 & 2 \\ 5 & 6 & 2 \end{vmatrix}+\begin{vmatrix} 7 & 0 & 3 \\ 9 & 2 & 3 \\ 5 & 1 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 3 & 2 \\ 3 & 2 & 3 \\ 6 & 1 & 2 \end{vmatrix}\right)\lambda^2$$

$$+ (-1)^1\left(\begin{vmatrix}-2 & 3 & 5 & 6\\ 9 & 7 & -1 & 0\\ -4 & -2 & 1 & 3\\ 5 & 9 & 3 & 2\end{vmatrix}+\begin{vmatrix}-2 & 3 & 5 & -1\\ 9 & 7 & -1 & 3\\ -4 & -2 & 1 & 2\\ 4 & 5 & 6 & 2\end{vmatrix}+\begin{vmatrix}-2 & 3 & 6 & -1\\ 9 & 7 & 0 & 3\\ 5 & 9 & 2 & 3\\ 4 & 5 & 1 & 2\end{vmatrix}+\begin{vmatrix}-2 & 5 & 6 & -1\\ -4 & 1 & 3 & 2\\ 5 & 3 & 2 & 3\\ 4 & 6 & 1 & 2\end{vmatrix}+\begin{vmatrix}7 & -1 & 0 & 3\\ -2 & 1 & 3 & 2\\ 9 & 3 & 2 & 3\\ 5 & 6 & 1 & 2\end{vmatrix}\right)\lambda^1+(-1)^0\begin{vmatrix}-2 & 3 & 5 & 6 & -1\\ 9 & 7 & -1 & 0 & 3\\ -4 & -2 & 1 & 3 & 2\\ 5 & 9 & 3 & 2 & 3\\ 4 & 5 & 6 & 1 & 2\end{vmatrix}\lambda^0= 0$$

Or $(-1)^5 \lambda^5 + (-1)^4 (10)\lambda^4+(-1)^3(-55)\lambda^3+(-1)^2(162)\lambda^2+(-1)^1(768)\lambda^1+(-1)^0(-3936)\lambda^0= 0$

Or $-\lambda^5 + 10\lambda^4+55\lambda^3+162\lambda^2-768\lambda^1-3936 = 0$

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